**Introduction to Analysis**
**VERY BIG Home Work ~~+ HW 10~~, due THE DAY OF FINAL EXAM.**
**Instructor: Prof. Artem Zvavitch**
**There are 22 problems 5 points each!**

**Problem 1.** *Use the principal of mathematical induction to show that for any numbers $b$ and $q \neq 1$ if*

$$x_i = b \cdot q^i,$$

*then*

$$\sum_{i=0}^{n} x_i = b\frac{q^{n+1} - 1}{q - 1}.$$

*Please, also find*

$$\lim_{n\to\infty} \sum_{i=0}^{n} x_i.$$

**Problem 2.** *Consider the set*

$$S = \{[x, y] \times [c, d], \text{where } x, y, c, d \text{ are rational numbers}\}.$$

*Show that $S$ is a countable set.*

**Problem 3.** *Prove that*

$$\sqrt{|a| + |b|} \leq \sqrt{|a|} + \sqrt{|b|}.$$

*Can you generalize this fact? I.e. find all $p$ for which*

$$(|a| + |b|)^p \leq |a|^p + |b|^p.$$

**Problem 4.** *Show that $|x - 5|$ is a continuous function on $R$. Is it uniformly continuous?*

**Problem 5.** *Show that $(x - 5)^2$ is a continuous function on $R$. Is it uniformly continuous?*

**Problem 6.** *Prove that*

$$\sup_{x\in A} |f(x) + g(x)| \leq \sup_{x\in A} |f(x)| + \sup_{x\in A} |g(x)|,$$

*and*

$$\sup_{x\in A} |f(x) \times g(x)| \leq \sup_{x\in A} |f(x)| \times \sup_{x\in A} |g(x)|.$$

**Problem 7.** *Prove that there exists a positive real number $x$ so that $x^2 = 5$. Please, also prove that $2 < x < 2.5$.*

*Please, find a sequence $\{x_n\}_{n=1}^{\infty}$ of rational numbers such that*

$$\lim_{n\to 1} x_n = \sqrt{5}.$$

**Problem 8.** *It it true that the sum of two rational numbers is rational?*

*Is it true that the sum of irrational number and rational number is necessary irrational?*

*Is it true that the sum of two irrational numbers is necessary irrational?*

**Do not forget to show a proof, if the answer to any of questions is Yes**

**Problem 9.** *Use the definition of limit to show that*

$$\lim_{n\to\infty} \frac{21+n}{-8-n} = -1$$

*and*

$$\lim_{n\to\infty} \frac{n^2+1}{n} = \infty.$$

**Problem 10.** *Show that*

$$a_n = \sum_{i=0}^{n} \frac{(-1)^i}{3^i}$$

*is a convergent sequence.*

**Problem 11.** *Show that if $a_{n+1} = a_n + n^{-n}$, then $a_n$ is a convergent sequence.*

**Problem 12.** *Let $x_1 = 0$ and $x_{n+1} = \frac{1}{4}(x_n + 1)$ show that a) $x_n$ is bounded. b) Monotone c) convergent. Finally find the limit of $x_n$.*

**Problem 13.** *Prove that if $a_n$ is a Cauchy sequence then $|a_n|$ is also a Cauchy sequence. Is it true that if $|b_n|$ is a Cauchy sequence then $b_n$ is a Cauchy sequence? Is it true that $b_n$ has a convergent subsequence?*

**Problem 14.** *Use $\varepsilon - \delta$ definition of limit to show that*

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$$

**Problem 15.** *Let $f(x)$ be a continuous function such that $f(2^{-n}) = \frac{n+1}{n}$ , find $f(0)$.*

**Problem 16.** *Show that*

$$\lim_{x\to 3} \frac{1}{x-3}$$

*does not exists.*

**Problem 17.** *Prove that if $f(x)$ and $g(x)$ are uniformly continuous functions on $A$, then $f(x) + g(x)$ is also uniformly continuous on $A$.*

**Problem 18.** *Let* $f(x)$ *be a continuous function on* $[a, b]$*, such that* $f(a) \times f(b) \leq 0$*, show that then there exist* $c \in [a, b]$ *such that* $f(c) = 0$*.*

**Problem 19.** *Let* $f(x)$ *be a continuous function on* $[0, 1]$ *such that* $f(0) = -1$ *and* $f(1) = 2$*, show that there exist* $x_0 \in [0, 1]$*, such that* $f(x_0) = x_0$*.*

**Problem 20.** *Show that* $q(x) = -3x^2$ *is a Lipschitz function on* $[0, 1]$*. Show that if* $f$ *is a Lipschitz function with Lipschitz constant* $L$*, then* $g(x) = -3f(x)$ *is a Lipschitz function with constant 3L.*

**Problem 21.** *Assume* $f(x)$ *and* $g(x)$ *are monotone increasing functions on* $[0, 1]$*, is it true that* $f(x) \times g(x)$ *is also monotone increasing?*

**Problem 22.** *Assume that* $f(x)$ *is a strictly increasing function on* $[a, b]$*, show that* $f^{-1}(x)$ *(inverse of* $f$*) is also an increasing function on* $[f(a), f(b)]$*.*