Functions of Real Variables 1 (62051/72051)
Final Home Work , due on December 13.
Each Problem is 17 points
Instructor: Prof. Artem Zvavitch.

Problem 1. *Let* $\{E_n\}_{n=1}^{\infty}$ *be a sequence of nonempty (Lebesgue) measurable subsets of* $[0,1]$ *satisfying*

$$\lim_{n\to\infty} m(E_n) = 1.$$

Show that for each $\varepsilon \in [0,1)$ *there exists a subsequence* $\{E_{n_k}\}_{k=1}^{\infty}$ *of* $\{E_n\}_{n=1}^{\infty}$ *such that*

$$m(\cap_{k=1}^{\infty} E_{n_k}) \geq \varepsilon$$

Problem 2. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be any function (not necessarily measurable!). Prove that the set of points* $x \in \mathbb{R}$ *such that*

$$F(y) \leq F(x) \leq F(z)$$

for all $y \leq x$ *and* $z \geq x$ *is a Borel set.*

Problem 3. *Suppose that* $f_n : \mathbb{R}^d \to \mathbb{R}$ *is a sequence of measurable function that converge to a measurable function* f *m-almost everywhere. In addition, suppose that there is a non-negative measurable function* F *such that*

$$\int_{\mathbb{R}^d} F dx < \infty \text{ and } |f_n| \leq F \text{ for all } n.$$

Prove that

$$\int_{\mathbb{R}^d} \limsup f_n dx \geq \limsup \int_{\mathbb{R}^d} f_n dx.$$

Also, please, give an example to show that the above conclusion may fail without the assumption of the existence of the integrable dominating function F.

Problem 4. *Consider a sequence for functions* $f_n : [0,2] \to \mathbb{R}$ *such that* $f(0) = 0$ *and* $f(x) = \frac{\sin x^n}{x^n}$ *for* $x \in (0,2]$. *Find*

$$\lim_{n\to\infty} \int\limits_{[0,2]} f_n(x) dx.$$

Problem 5. *Suppose that* $A \subset \mathbb{R}$ *satisfies* $m_1(A) = 0$, *where* m_1 *denotes the one-dimensional Lebesque measure. Suppose* $f : \mathbb{R} \to \mathbb{R}^2$ *satisfies*

$$|f(x) - f(y)| \leq \sqrt{|x-y|}, \text{ for every } x, y \in \mathbb{R}.$$

Show that $m_2(f(A)) = 0$, m_2 *denotes the two-dimensional Lebesque measure on* $\mathbb{R}^2$.

Problem 6. *Let* f *be absolutely continuous in the interval* $[\varepsilon, 1]$ *for each* $0 < \varepsilon < 1$. *Does the continuity off* f *at 0 imply that* f *is absolutely continuous on* $[0,1]$*? What if* f *is also of bounded variation on* $[0,1]$*?*

Problem 7. *Show that the case of equality in the Cauchy-Schwarz inequality (i.e.* $|(f,g)| = \|f\|\|g\|$*) is possible if and only if* $f = cg$ *for some* $c \in \mathbb{R}$.

Problem 8. *Use the definition of* $\ell_2(\mathbb{Z})$ *to prove that this space is complete and separable.*

Problem 9. *Give an example of a function* $f \in L_2(\mathbb{R}^d)$ *but* $f \notin L_1(\mathbb{R}^d)$. *Also give an example of a function* $g \in L^1(\mathbb{R}^d)$ *but* $g \notin L_2(\mathbb{R}^d)$. *Also prove that if* $f(x)$ *is a bounded function in* $L_1(\mathbb{R}^d)$ *then* $f \in L_2(\mathbb{R}^d)$.

Problem 10. *Prove that simple functions and continuous functions of compact support are dense subspaces of* $L_2(\mathbb{R}^d)$.