

**Functions of Real Variables 1 (62051/72051)**

**Final Home Work , due on December 17.**

**Each Problem is 13 points**

**Instructor: Prof. Artem Zvavitch.**

**Problem 1.** Let  $\{E_n\}_{n=1}^\infty$  be a sequence of nonempty (Lebesgue) measurable subsets of  $[0, 1]$  satisfying

$$\lim_{n \rightarrow \infty} m(E_n) = 1.$$

Show that for each  $\varepsilon \in [0, 1)$  there exists a subsequence  $\{E_{n_k}\}_{k=1}^\infty$  of  $\{E_n\}_{n=1}^\infty$  such that

$$m(\cap_{k=1}^\infty E_{n_k}) \geq \varepsilon$$

**Problem 2.** Suppose that  $f_n : \mathbb{R}^d \rightarrow \mathbb{R}$  is a sequence of measurable function that converge to a measurable function  $f$   $m$ -almost everywhere. In addition, suppose that there is a non-negative measurable function  $F$  such that

$$\int_{\mathbb{R}^d} F dx < \infty \text{ and } |f_n| \leq F \text{ for all } n.$$

Prove that

$$\int_{\mathbb{R}^d} \limsup f_n dx \geq \limsup \int_{\mathbb{R}^d} f_n dx.$$

Also, please, give an example to show that the above conclusion may fail without the assumption of the existence of the integrable dominating function  $F$ .

**Problem 3.** Consider a sequence for functions  $f_n : [0, 2] \rightarrow \mathbb{R}$  such that  $f_n(0) = 0$  and  $f_n(x) = \frac{\sin x^n}{x^n}$  for  $x \in (0, 2]$ . Find

$$\lim_{n \rightarrow \infty} \int_{[0,2]} f_n(x) dx.$$

**Problem 4.** Suppose that  $A \subset \mathbb{R}$  satisfies  $m_1(A) = 0$ , where  $m_1$  denotes the one-dimensional Lebesgue measure. Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}^2$  satisfies

$$|f(x) - f(y)| \leq \sqrt{|x - y|}, \text{ for every } x, y \in \mathbb{R}.$$

Show that  $m_2(f(A)) = 0$ ,  $m_2$  denotes the two-dimensional Lebesgue measure on  $\mathbb{R}^2$ .

**Problem 5.** Let  $f$  be absolutely continuous in the interval  $[\varepsilon, 1]$  for each  $0 < \varepsilon < 1$ . Does the continuity of  $f$  at 0 imply that  $f$  is absolutely continuous on  $[0, 1]$ ? What if  $f$  is also of bounded variation on  $[0, 1]$ ?

**Problem 6.** Prove the following variant of the **Vitali covering lemma**: If  $E$  is covered in the Vitali sense by a family  $\mathcal{B}$  of balls, and  $m_*(E) \in (0, \infty)$ , then for every  $\nu > 0$  there exists a disjoint collection of balls  $\{B_j\}_{j=1}^\infty$  in  $\mathcal{B}$  such that

$$m_*(E \setminus \bigcup_{j=1}^\infty B_j) = 0 \text{ and } \sum_{j=1}^\infty |B_j| \leq (1 + \nu)m_*(E).$$

**Problem 7.** Construct an increasing function on  $\mathbb{R}$  whose set of discontinuities is precisely  $\mathbb{Q}$ .

**Problem 8.** Suppose  $F$  is an increasing function on  $[a, b]$ .

- Prove that we can decompose  $F$  as  $F = F_A + F_C + F_J$ , where  $F_A, F_C, F_J$  are increasing,  $F_A$  is absolutely continuous;  $F_C$  is continuous and  $F_J$  is a jump function.
- Please, also show that  $F_A, F_C$  and  $F_J$  are uniquely determined for each  $F$  up to an additive constant.

Note, that the above construction is the Lebesgue decomposition of  $F$ .