

Analysis II (42001/52001)
Home Work 10, due Wednesday, April 12
Instructor: Prof. Artem Zvavitch

Problem 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that there exists $K \in \mathbb{R}$ such that $f(x) = Ke^x$ for all $x \in \mathbb{R}$.

Problem 2. Prove the Arithmetic-Geometric Mean inequality: for any positive numbers a_1, \dots, a_n we get

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Hint: Let $A = \frac{a_1 + \cdots + a_n}{n}$ and let $x_k = a_k/A - 1$. Plug x_k into inequality $1 + x \leq e^x$. Multiply the resulting inequalities to finish the proof.

Problem 3. Consider twice differentiable function $g(x) : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g''(x) = g(x) \text{ for all } x \in \mathbb{R}$$

and $g(0) = g'(0) = 0$. Prove that $g(x) = 0$.

Problem 4. Consider equation $f''(x) = f(x)$ for all $x \in \mathbb{R}$:

- Show that e^x and e^{-x} are solutions for this equation and thus hyperbolic cosine $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and hyperbolic sine $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ are solutions.
- Prove that any solution of the above equation can be written as

$$f(x) = \alpha \cosh(x) + \beta \sinh(x), \text{ for some } \alpha, \beta \in \mathbb{R}.$$

- Prove that $\cosh(x)$ is even and $\sinh(x)$ is odd functions.
- Prove that

$$\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y).$$

$$\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y).$$