Analysis 2 (42001/52001)
Final Home Work, due the day of the final exam, May 9:
14 problems, each problem 3 points. Instructor: Prof. Artem Zvavitch

Problem 1. *Consider function*

$$f(x) = 2x^4 + x^4 \sin(\frac{1}{x}) \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

Show that f has an absolute minimum at $x = 0$, but the derivative has both positive and negative values in every neighborhood of 0.

Problem 2. *Let $f(t) = signt$, i.e. $f(t) = -1$ for $t < 0$, $f(0) = 0$ and $f(t) = 1$ for $t > 0$, find*

$$\int_{-1}^{x} f(t)dt$$

Problem 3. *Prove that for $|x| < 1$*

$$\left|\sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| \leq \frac{1}{5040}.$$

Problem 4. *Show that the sequence $f_n(x) = x^n$ converges pointwise on $[0, 1]$ also, prove that the same sequence does not converges uniformly on this interval. Describe convergence of $f_n'(x)$, $\int_0^1 f_n(x)dx$ and $\int_0^t f_n(x)dx$, where $t \in [0, 1)$.*

Problem 5. *Use definition and properties of $\cos x$ and $\sin x$ functions to show that*

$$\sin(x + y) = \sin x \cos y + \sin y \cos x.$$

Problem 6. *Use definition and properties of $\cos x$ to show that*

$$\cos(\frac{1}{2}) \geq \frac{7}{8}.$$

Problem 7. *Discuss convergence or divergence of the following series:*

$$\sum_{i=1}^{\infty} 3^{-n} n!,$$

$$\sum_{i=1}^{\infty} (-1)^n (1 - \frac{1}{n})^n.$$

Problem 8. *Find the radius of convergence of the following series*

$$\sum_{n=1}^{\infty} (-2)^n \frac{x^n}{n^3}.$$

Problem 9. *Let $f(x) = \sum a_n x^n$ for $|x| < R$ and $f(x)$ is an even function (i.e. $f(x) = f(-x)$ for all $|x| < R$) show that $a_n = 0$ for all odd indexes n.*

Problem 10. *Please, give an example of a function which is infinitely differentiable on $\mathbb{R}$ (i.e. $f^{(n)}(x)$ exists for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$), such that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, but $f(x) \not\equiv 0$. Explain why this example do not contradict Teylor's theorem.*

Problem 11. *Let*
$$f(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}.$$
Prove that
$$\int_0^{\pi/2} f(x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

Problem 12. *Find the largest subset of* $\mathbb{R}$ *on which* $\sum_{k=1}^{\infty} \frac{x^{3k}}{k}$ *is convergent.*

Problem 13. *Find the Taylor series expansion around zero of* $Arctan(x)$*, find the radius of convergence of this series.*

Problem 14. *Find a series expansion for* $\int_0^x e^{-t^2} dt$.