Introduction to Analysis II
Home Work 1, due Wednesday, January 25.
Instructor: Prof. Artem Zvavitch

Problem 1. *Construct an example of a continuous function $f(x)$ which is not differentiable at points $x = -1$, $x = 1$, $x = 3$ and is differentiable everywhere else.*

Problem 2. *Construct an example of a differentiable non-constant function $f(x)$ such that $f'(-1) = f'(0) = f'(1) = 0$.*

Problem 3. *Construct an example of continuous functions $f(x), g(x)$ and $h(x)$ such that $f(x) = g(x)h(x)$ and $f(x)$ and $g(x)$ are differentiable at point $x = 0$ but $h(x)$ is not differentiable at that point.*

Problem 4. *Is it true that if $f'(x) = g'(x)$ for all $x \in \mathbb{R}$ then $f(x) = g(x)$ for each $x \in \mathbb{R}$? Is it true that if $f'(x) < g'(x)$ for each $x \in \mathbb{R}$ then $f(x) \leq g(x)$ for all $x \in \mathbb{R}$?*

Problem 5. *Consider a differentiable even (i.e. $f(x) = f(-x)$) function $f(x)$. Show that $f'(0) = 0$*

Problem 6. *Check if the function $f(x) = x \cdot |x|$ is differentiable at 0.*

Problem 7. *Prove that*

$$|\sin x - \sin y| \leq |x - y|, \text{ for all } x, y \in \mathbb{R}.$$

also prove that

$$|e^x - e^y| \geq |x - y|, \text{ for all } x, y \geq 0.$$

Definition: We say that the function $f : I \to \mathbb{R}$ is Lipschitz, if there exists constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \text{ for all } x, y \in I.$$

Problem 8. *Consider an interval $I = [a, b]$ and a differentiable function f on I. Assume that $f'(x)$ is is bounded on I, prove that f is a Lipschitz function on I.*

Problem 9. *Consider a Lipschitz function f on $I = [a, b]$, is it true that then f is differentiable on I.*

Problem 10. *Find an example of a function which is differentiable on $I = [-1, 1]$ but not twice differentiable on I.*