Introduction to Analysis II
Home Work 5, due Wednesday, February 22 .
Instructor: Prof. Artem Zvavitch

Problem 1. *Use the definition of the Riemann integral to show that the function* $f(x) : [0,2] \to \mathbb{R}$ *such that* $f(x) = 3x$ *for* $x \in [0,1]$ *and* 7 *otherwise is Riemann integrable on* $[0,2]$.

Problem 2. *Consider Riemann integrable function* f *on* $[a,b]$ *such that* $|f(x)| < M$ *for all* $x \in [a,b]$ *show that then*

$$|\int_a^b f dx| \le M(b-a).$$

Problem 3. *Consider function* h *defined by* $h(x) = x+1$ *for* $x \in [0,1]$ *rational, and* $h(x) = 0$ *otherwise. Show that* h *is not Riemann integrable on* $[0,1]$.

Problem 4. *Consider a continuous function* $f(x) \ge 0$ *such that*

$$\int_a^b f = 0.$$

Prove that $f(x) = 0$ *for all* $x \in [a,b]$.

Problem 5. *Is it possible to remove continuity assumption in the previous problem?*

Problem 6. (BONUS 10 pts) *For* $x \in [0,5]$, *find a good bound for the error which comes from estimating* $\ln(1+x)$ *by* $x - \frac{x^2}{2}$.
HINT: *use Taylors formula: if* $f(x) = \ln(1+x)$

$$\ln(1+x) = x + \frac{x^2}{2} + \frac{f'''(c)}{3!}(x-c)^3$$

where $x, c \in [0,5]$.

Problem 7. (BONUS 10 pts) *Let function* $f : \mathbb{R} \to \mathbb{R}$ *be twice differentiable and convex. Show that for any* $c \in \mathbb{R}$ *the graph of* f *is never below the tangent line to the graph* f *at point* $(c, f(c))$, *more precisely, prove that for any* $x, c \in \mathbb{R}$:

$$f(x) \ge f'(c)(x-c) + f(c).$$

HINT: *Again use Taylors formula:*

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(a)}{2!}(x-a)^2$$

where a *belongs to the interval between* x *and* a, *next use that* f *is convex.*