Analysis II (42001/52001)
HW8, due Wednesday, March 15
Instructor: Prof. Artem Zvavitch

Problem 1 *Consider $x > 0$. Prove that*

$$\int_0^x t^2 e^{-\frac{t^2}{2}} dt \le x(1 - e^{-\frac{x^2}{2}}).$$

Problem 2 *Show that the function*

$$f(x) = \int_0^x e^{-t^2} dt$$

is a Lipschitz function.

Problem 3 *Find all continuous functions f such that*

$$\int_0^x f(t)dt = \int_0^{2x} f(t)dt, \text{ for all } x \in \mathbb{R}.$$

Problem 4 *Consider functions $f, g \in \mathcal{R}[a, b]$*

- *Show that $\int\limits_a^b (tf \pm \frac{1}{t}g)^2 dx \ge 0$ for all $t \in \mathbb{R}$.*
- *Use previous fact to show that*

$$2\left|\int_a^b (fg)\right| \le t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2.$$

- *If $\int f^2 = 0$, show that $\int_a^b fg = 0$.*
- *Now, please, prove one of the most famous inequalities in Analysis* **Cauch-Bunyakowski-Schwartz Inequality***:*

$$\left(\int_a^b |fg|\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

- *Finally, prove the triangle inequality for functions*

$$\sqrt{\int_a^b (f-g)^2} \le \sqrt{\int_a^b f^2} + \sqrt{\int_a^b g^2}.$$