Analysis II (42002/52002)
Exam 2/ PREPARATION and HW 9
Instructor: Prof. Artem Zvavitch
due Wednesday, March 22.
50 points - yes 30 are bonus points!

Problem 1. *Give an example of function* $f \in \mathcal{R}[-1,1]$ *such that* $F(x) = \int_{-1}^{x} f(t)dt$ *is not differentiable function at* $x = 0$.

Problem 2. *Give an upper and lower bounds for the following integral*

$$\int_0^{\pi} e^{\sin^2 x} dx.$$

Problem 3. *Consider function* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$\int_{-1}^{1} f^2(x)dx = 0,$$

Is it true that then $f(x)$ *is equal to zero on* $[-1,1]$*? Will your answer change (and why) if we assume that* $f(x)$ *is a continuos function.*

Problem 4. *Suppose that* f *is uniformly continuous function on* $\mathbb{R}$. *Consider a sequence* $\{y_n\} \in \mathbb{R}$ *such*

$$\lim_{n\to\infty} y_n = 0.$$

Let $f_n = f(x+y_n)$. *Show that* $f_n(x)$ *converges uniformly on* $\mathbb{R}$ *to* $f(x)$.

Problem 5. *Show that the sequence* $f_n(x) = nxe^{-nx^2/2}$ *converges uniformly on* (a,∞), *for any* $a > 0$, *but it does not converge uniformly on* $[0,\infty)$.

Problem 6. *Prove that if sequences* f_n, g_n *converge uniformly on* A *to* f *and* g, *resectively, then* $f_n + g_n$ *converges uniformly on* A *to* $f + g$.

Problem 7. *Show that the product of uniformly convergent sequences is not necessary uniformly convergent.* **Hint:** *consider* $f_n = x + \frac{1}{n}$ *and* f_n^2 *on* $\mathbb{R}$.

Problem 8. *Now let's repair the product property: Prove that if sequences of* **bounded functions** f_n, g_n *converge uniformly on* A *to* f *and* g, *resectively, then* $f_n g_n$ *converges uniformly on* A *to* fg.

Problem 9. *Show that the sequence* $x^n/(1+x^n)$ *dooes not converge uniformly on* $[0,2]$ *by showing that the limit function is not continuous on* $[0,2]$.

Problem 10. *Let* $f_n(x) = e^{-nx}/n$ *for* $x \geq 0$*. Study the relation between* $\lim f_n$ *and* $\lim f_n'$*.*

Problem 11. *Assume that* $f_n(x)$ *converges uniformly to* $f(x)$ *on* $[a, b]$*, prove that then*

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)| = 0.$$

Show that converse is not true.

Problem 12. *Consider a sequence of bounded functions* $f_n(x)$*. If* $f_n(x)$ *converges pointwise to* $f(x)$ *is it true that* $f(x)$ *is a bounded function? if If* $f_n(x)$ *converges uniformly to* $f(x)$ *is it true that* $f(x)$ *is a bounded function?*

Problem 13. *Find a sequence of continuous functions convergent pointwise to function* $f(x) = \operatorname{sign}(x)$*.*