

**Functions of Real Variables 1 (62051/72051)**  
**Home Work 2, due on Monday, September 9.**  
**Instructor: Prof. Artem Zvavitch.**

**Problem 1.** Let  $C \in [0, 1]$  be Cantor set.

- Is it true that  $C$  contain irrational points?
- Is it true that  $C$  contains only boundary points (of  $C$ )?
- Please, find  $\bar{C}$  (closure of  $C$ ).

**Problem 2.** Consider the unit interval  $[0, 1]$ , and let  $\xi$  be fixed real number with  $\xi \in (0, 1)$  (note that the case  $\xi = 1/3$  corresponds to the regular Cantor set we learned in our lectures).

In stage 1 of the construction, remove the centrally situated open interval in  $[0, 1]$  of length  $\xi$ . In stage 2 remove the centrally situated open intervals each of **relative** length  $\xi$  (i.e. if the interval has length  $a$  you remove an interval of length  $\xi \times a$ ), one in each of the remaining intervals after stage 1, and so on.

Let  $C_\xi$  denote the set which remains after applying the above procedure indefinitely.

- Prove that  $C_\xi$  is compact.
- Prove that  $C_\xi$  is totally disconnected and perfect.
- Actually, prove that the complement of  $C_\xi$  in  $[0, 1]$  is the union of open intervals of total length equal to 1.
- Show directly that  $m_*(C_\xi) = 0$ .

**Problem 3.** We may also define the outer measure by taking coverings by rectangles instead of cubes. More precisely, we define

$$m_*^R(E) = \inf \sum_{j=1}^{\infty} |R_j|,$$

where the inf is now taken over all countable coverings  $E \subset \bigcup_{j=1}^{\infty} R_j$  by closed rectangles. Show that actually

$$m_*^R(E) = m_*(E),$$

for every subset  $E$  in  $\mathbb{R}^d$ .

**Problem 4.** Suppose  $E$  is a given set, and  $O_n$ , for  $n \in \mathbb{N}$ , is the set defined by

$$O_n = \{x \in \mathbb{R}^d : d(x, E) < \frac{1}{n}\}.$$

- Prove that  $O_n$  is open.
- Prove that if  $E$  is compact, then  $m(E) = \lim_{n \rightarrow \infty} m(O_n)$ .
- Would the above be true for  $E$  closed and unbounded set?
- Would the above be true for  $E$  open and bounded set?