Functions of Real Variables 1 (62051/72051)
Home Work 4, due on Wednesday October 2.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Let $\mathcal{N}$ denote the non-measurable subset of $[0,1]$, constructed in class and in the book.*

- *Prove that if E is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.*
- *Assume that G is a subset of $\mathbb{R}$ with $m_*(G) > 0$, prove that there is a subset of G such that it is non-measurable.*
- *Prove that if $\mathcal{N}^c = [0,1] \setminus \mathcal{N}$, then $m_*(\mathcal{N}^c) = 1$.*
- *Now, conclude that*

$$m_*(\mathcal{N}) + m_*(\mathcal{N}^c) \neq m_*(\mathcal{N} \cup \mathcal{N}^c).$$

Problem 2. *Let C_ξ and C_ν be two Cantor sets (constructed in previous HW). Show that there exist a function $F : [0,1] \to [0,1]$ with the following properties*

- *F is continuous and bijective.*
- *F is monotonically increasing.*
- *F maps C_ξ surjectively onto C_ν.*
- *Now give an example of a measurable function f and a continuous function Φ so that $f \circ \Phi$ is non-measurable. One may use function F constructed above (BUT YOU NEED TO PLAY WITH IT). One of the ideas is to take two measurable sets C_1 and C_2 such that $m(C_1) > 0$ but $m(C_2) = 0$ and function $\Phi : C_1 \to C_2$, continuous. Also take $\mathcal{N} \subset C_1$ - non-measurable set and define $f = \chi_{\Phi(\mathcal{N})}$.*
- *Use the above construction to show that there exists a Lebesque measurable set that is not a Borel set.*

Problem 3. *Prove that there is a continuous function that maps a Lebegue measurable set to non-measurable set. The previous problem may help!*

Problem 4. *Suppose that Γ is a curve $(x, f(x))$ in $\mathbb{R}^2$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that $m(\Gamma) = 0$.*

Problem 5. (The Borel-Cantelli lemma): *Suppose $\{E_k\}_{k=1}^{\infty}$ is countable family of measurable subsets of $\mathbb{R}^d$ and that*

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\} = \limsup_{k\to\infty} (E_k).$$

- *Show that E is measurable.*
- *Prove that $m(E) = 0$.*

Hint: *write $E = \cap_{n=1}^{\infty} \cup_{k \geq n} E_k$.*

Problem 6. *Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for a.e. x. Show that there exists a sequence c_n of positive real numbers such that*

$$\frac{f_n(x)}{c_n} \to 0, \ \text{a.e. } x.$$

Hint: *Pick c_n such that $m(\{x : |f_n(x)/c_n| > 1/n\}) < 2^{-n}$, and apply the Borel-Cantelli lemma.*