Functions of Real Variables 2 (62052/72052)
Home Work 1 (are you ready for this class?), due on Wednesday January 25.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Prove the following elementary version of a Vitali covering lemma: Suppose that* $\mathcal{B} = \{B_1, B_2, \dots, B_N\}$ *is a finite collection of open Euclidean balls in* $\mathbb{R}^d$*. Then there exists a disjoint sub-collection* $B_{i_1}, B_{i_2}, \dots, B_{i_k}$ *of* $\mathcal{B}$ *that satisfies*

$$m(\bigcup_{l=1}^{N} B_l) \leq 3^d \sum_{j=1}^{k} m(B_{i_k}).$$

Problem 2. *Suppose that, for each* $j \in \mathbb{N}$, $f_j : [0,1] \to \mathbb{R}$ *is a Lebesgue measurable function such that* $0 \leq f_j \leq \frac{3}{2}$ *and*

$$\int_0^1 f_j dm = 1.$$

Prove that

$$m\left(\{x \in [0,1] : \limsup_{j\to\infty} f_j(x) \geq \frac{1}{2}\}\right) \geq \frac{1}{2}.$$

Problem 3. *Consider a sequence for functions* $f_n : [0,2] \to \mathbb{R}$ *such that* $f(0) = 0$ *and* $f(x) = \frac{\sin x^n}{x^n}$ *for* $x \in (0,2]$*. Find*

$$\lim_{n\to\infty} \int\limits_{[0,2]} f_n(x)dx.$$

Problem 4. *Show that if* $f : \mathbb{R} \to \mathbb{R}$ *is absolutely continuous, then* f *maps sets of measure zero to sets of measure zero.*

Problem 5. *The operator* $T : H \to H$ *is an isometry if* $\|Tf\| = \|f\|$ *for all* $f \in H$*.*

- *Please, prove that if* T *is an isometry then* $(Tf, Tg) = (f, g)$ *for all* $f, g \in H$*.*
- *Now prove that if* T *is an isometry then* $T^*T = I$*.*
- *Now prove that if* T *is surjective and isometry (and thus unitary) then* $TT^* = I$*.*
- *Give an example of an isometry* T *that is not unitary.* **Hint:** *consider* $\ell_2(\mathbb{N})$ *and the map which takes* $(a_1, a_2, \dots)$ *to* $(0, a_1, a_2, \dots)$*.*
- *Now Prove that if* $T * T$ *is unitary then* T *is an isometry.* **Hint:** *Start with*

  $$\|Tf\|^2 = (f, T^*Tf)$$

  and use Holders inequality to get $\|Tf\| \leq \|f\|$*. Next consider* $\|f\| = \|T^*Tf\|$ *do get the opposite inequality.*