

**Functions of Real Variables 1 (62051/72051)**  
**Home Work 1, due on Tuesday SEPTEMBER 3.**  
**Instructor: Prof. Artem Zvavitch.**

**Problem 1.** Please, answer the following questions. Please, do not forget that you must explain your answers!! All sets below are subsets of  $\mathbb{R}^d$ .

- Assume  $A_1, A_2, \dots, A_i, \dots$  are open sets and  $N$  is a fixed natural number. What can you say about

(1)  $\bigcap_{i=1}^N A_i$ ?

(2)  $\bigcup_{i=1}^N A_i$ ?

(3)  $\bigcap_{i=1}^{\infty} A_i$ ?

(4)  $\bigcup_{i=1}^{\infty} A_i$ ?

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(4)  $\bigcup_{i=1}^{\infty} A_i$ ?

- Assume  $A$  and  $B$  are open sets, what can you say about set  $A + B$ , where

$$A + B = \{a + b : a \in A, b \in B\}.$$

- Assume  $A$  and  $B$  are closed sets, what can you say about set  $A + B$ ,
- Assume  $A$  is an open set what can you say about set  $P_{e_1} A$ , where  $P_{e_1}(A)$  is an orthogonal projection of  $A$  onto subspace orthogonal to the first basis vector  $e_1$ , i.e.

$$P_{e_1}(A) = \{(0, a_2, \dots, a_d) : (a_1, a_2, \dots, a_d) \in A\}.$$

- Is it true that a subset of a compact set is compact?
- Assume  $A$  is a compact set, what can you say about  $P_{e_1} A$ ?

**Problem 2.** Is it true that every open set in  $\mathbb{R}^2$  can be written as a disjoint union of open rectangles? (**Hint:** play with a unit disc. What about boundary of the rectangles?)

**Problem 3.** Let  $A$  and  $B$  be closed rectangles in  $\mathbb{R}^d$ , prove that

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}.$$

**Hint:** you may need to use some classical inequality for real numbers, you do not need to prove it, just state it and provide a reference.

**Problem 4.** Let  $C \in [0, 1]$  be Cantor set.

- Is it true that  $C$  contain irrational points?
- Is it true that  $C$  contains only boundary points (of  $C$ )?
- Please, find  $\bar{C}$  (closure of  $C$ ).

**Problem 5.** Consider the unit interval  $[0, 1]$ , and let  $\xi$  be fixed real number with  $\xi \in (0, 1)$  (note that the case  $\xi = 1/3$  corresponds to the regular Cantor set we learned in our lectures).

In stage 1 of the construction, remove the centrally situated open interval in  $[0, 1]$  of length  $\xi$ . In stage 2 remove the centrally situated open intervals each of **relative** length  $\xi$  (i.e. if the interval has length  $a$  you remove an interval of length  $\xi \times a$ ), one in each of the remaining intervals after stage 1, and so on.

Let  $C_\xi$  denote the set which remains after applying the above procedure indefinitely.

- Prove that  $C_\xi$  is compact.
- Prove that  $C_\xi$  is totally disconnected and perfect.
- Actually, prove that the complement of  $C_\xi$  in  $[0, 1]$  is the union of open intervals of total length equal to 1.
- Show directly that  $m_*(C_\xi) = 0$ .