**Functions of Real Variables 2 (62052/72052)**
**Home Work 1, due on Friday January 31.**
**Instructor: Prof. Artem Zvavitch.**

**Problem 1.** *Prove from the definition that* $\ell^2(\mathbb{N})$ *is complete and separable.*

**Problem 2.** *Suppose* $\{g_k\}_{k=1}^{\infty}$ *is an orthonormal basis for* $L_2(\mathbb{R}^d)$*. Prove that the collection* $\{g_{k,j}\}_{1\le k,j\le\infty}$*, where*

$$g_{k,j}(x,y) = g_k(x)g_j(y).$$

*is an orthonormal basis for* $L_2(\mathbb{R}^{2d})$*.* **Suggestion:** *Note that* $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$ *and use the Fubini theorem to verify the orthonormality. Next consider the case* $(f, g_{k,j}) = 0$ *for all* $k, j$ *and again use the Fubini theorem.*

**Problem 3.** *Let* $\nu(t)$ *be a fixed, continuous strictly positive function on* $[a, b]$*. Define* $L_2([a, b], \nu)$ *to be the space of all measurable functions* $f$ *on* $[a, b]$ *such that*

$$\int_a^b |f(t)|^2 \nu(t)dt < \infty.$$

*Define the inner product on* $L_2([a, b], \nu)$ *as*

$$\int_a^b f(t)\overline{g(t)}\nu(t)dt.$$

*Prove that* $L_2([a, b], \nu)$ *is a Hilbert space and find the mapping which gives a unitary correspondence between* $L_2([a, b], \nu)$ *and* $L_2([a, b])$*.*

**Problem 4.** *Let* $S$ *be (not necessary closed) subspace of a Hilber space* $H$*. Prove that* $(S^{\perp})^{\perp}$ *is the smallest closed subspace of* $H$ *that contains* $S$*.*

**Problem 5.** *Let* $P$ *be the orthogonal projection associated with a closed subspace* $S$ *in a Hilbert space* $H$*, that is* $P$ *is a linear operator such that*

$$P(f) = f \text{ if } f \in S \text{ and } P(f) = 0 \text{ if } f \in S^{\perp}.$$

- *Show that* $P^2 = P$ *and* $P^* = P$*.*
- *Conversely, if* $P$ *is any bounded operator satisfying* $P^2 = P$ *and* $P^* = P$*, prove that* $P$ *is the orthogonal projection for some closed subspace of* $H$*.*
- *Using the ideas of orthogonal projection operator* $P$ *prove that if* $S$ *is a closed subspace of a separable Hilbert space, then* $S$ *is also a separable Hilbert space.*

**Problem 6.** *The operator* $T : H \to H$ *is an isometry if* $\|Tf\| = \|f\|$ *for all* $f \in H$*.*

- *Please, prove that if* $T$ *is an isometry then* $(Tf, Tg) = (f, g)$ *for all* $f, g \in H$*.*
- *Now prove that if* $T$ *is an isometry then* $T^*T = I$*.*
- *Now prove that if* $T$ *is surjective and isometry (and thus unitary) then* $TT^* = I$*.*
- *Give an example of an isometry* $T$ *that is not unitary.* **Hint:** *consider* $\ell_2(\mathbb{N})$ *and the map which takes* $(a_1, a_2, \dots)$ *to* $(0, a_1, a_2, \dots)$*.*
- *Now Prove that if* $T^*T$ *is unitary then* $T$ *is an isometry.* **Hint:** *Start with*

  $$\|Tf\|^2 = (f, T^*Tf)$$

  *and use Hölder's inequality to get* $\|Tf\| \le \|f\|$*. Next consider* $\|f\| = \|T^*Tf\|$ *do get the opposite inequality.*