Functions of Real Variables 1 (62051/72051)
Home Work 1, due on Wednesday, Augist 30.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Please, answer the following questions. Please, do not forget that you must explain your answers!! All sets below are subsets of* $\mathbb{R}^d$.

- *Assume* $A_1, A_2, \dots, A_i, \dots$ *are open sets and* N *is a fixed natural number. What can you say about*
  (1) $\bigcap_{i=1}^{N} A_i$?
  (2) $\bigcup_{i=1}^{N} A_i$?
  (3) $\bigcap_{i=1}^{\infty} A_i$?
  (4) $\bigcup_{i=1}^{\infty} A_i$?
- *Assume* $A_1, A_2, \dots, A_i, \dots$ *are closed sets. What can you say about*
  (1) $\bigcap_{i=1}^{N} A_i$?
  (2) $\bigcup_{i=1}^{N} A_i$?
  (3) $\bigcap_{i=1}^{\infty} A_i$?
  (4) $\bigcup_{i=1}^{\infty} A_i$?
- *Assume* A *and* B *are open sets in* $\mathbb{R}^d$, *what can you say about set* $A+B$, *where*
$$A + B = \{a + b : a \in A, b \in B\}.$$
- *Assume* A *and* B *are closed sets in* $\mathbb{R}^d$, *what can you say about set* $A+B$ *(BE CAREFUL!).*
- *Assume* A *in* $\mathbb{R}^d$ *is an open set what can you say about set* $P_{e_1}A$, *where* $P_{e_1}(A)$ *is an orthogonal projection of* A *onto subspace orthogonal to the first basis vector* e_1, *i.e.*
$$P_{e_1}(A) = \{(0, a_2, \dots, a_d) : (a_1, a_2, \dots, a_d) \in A\}.$$
Note that here we can ask two questions: Is $P_{e_1}(A)$ *is an open set in* $\mathbb{R}^d$*? Is* $P_{e_1}(A)$ *is an open set in subspace orthogonal to the first basis vector* e_1*?*
- *Is it true that a subset of a compact set is compact?*
- *Assume* A *is a compact set, what can you say about* $P_{e_1}A$ *?*

Problem 2. *Is it true that every open set in* $\mathbb{R}^2$ *can be written as a disjoint union of open rectangles? (***Hint:** *play with a unit disc. What about boundary of the rectangles?*

Problem 3. *Let* A *and* B *be closed rectangles in* $\mathbb{R}^d$, *prove that*
$$|A + B|^{\frac{1}{d}} \ge |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}.$$

Hint: *you may need to use some classical inequality for real numbers, you do not need to prove it, just stay it and provide a reference.*