Functions of Real Variables 2 (62051/72051)
Home Work 1, due on Wednesday SEPTEMBER 14.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Assume A and B are closed sets, what can you say about set $A+B$, where*

$$A+B=\{a+b : a\in A, b\in B\}.$$

Problem 2. *Let $C\in[0,1]$ be Cantor set.*

- *Is it true that C contain irrational points?*
- *Is it true that C contains only boundary points (of C)?*
- *Please, find $\bar{C}$ (closure of C).*

Problem 3. *Consider the unit interval $[0,1]$, and let ξ be fixed real number with $\xi\in(0,1)$ (note that the case $\xi=1/3$ corresponds to the regular Cantor set we learned in our lectures).*

In stage 1 of the construction, remove the centrally situated open interval in $[0,1]$ of length ξ. In stage 2 remove the centrally situated open intervals each of **relative** *length ξ (i.e. if the interval has length a you remove an interval of length $\xi\times a$), one in each of the remaining intervals after stage 1, and so on.*

Let C_ξ denote the set which remains after applying the above procedure indefinitely.

- *Prove that C_ξ is compact.*
- *Prove that C_ξ is totally disconnected and perfect.*
- *Atually, prove that the complement of C_ξ in $[0,1]$ is the union of open intervals of total length equal to 1.*
- *Show directly that $m_*(C_\xi)=0$.*

Problem 4. *We may also define the outer measure by taking coverings by rectangles instead of cubes. More precisely, we define*

$$m_*^R(E)=\inf\sum_{j=1}^{\infty}|R_j|,$$

where the $\inf$ *is now taken over all countable coverings $E\subset\bigcup_{j=1}^{\infty}R_j$ by closed rectangles. Show that actually*

$$m_*^R(E)=m_*(E),$$

for every subster E in $\mathbb{R}^d$.