

**Functions of Real Variables 2 (62052/72052)**  
**Home Work 2, due on Wednesday, February 1.**  
**Instructor: Prof. Artem Zvavitch.**

**Problem 1.** Let  $T : H_1 \rightarrow H_2$  be a linear operator, from finite dimensional Hilbert Space  $H_1$  to a Hilbert space  $H_2$ . Prove that  $T$  is bounded.

**Problem 2.** Assume  $T$  is a bounded linear operator on a Hilbert space, prove that

$$\|TT^*\| = \|T^*T\| = \|T\|^2 = \|T^*\|^2$$

**Problem 3.** Assume  $H$  is an infinite-dimensional Hilbert space. Give an example of a sequence  $\{f_n\}$  in  $H$ , with  $\|f_n\| = 1$  for all  $n$ , such that  $\{f_n\}$  has no converging subsequence. Next show that for any sequence  $\{f_n\}$  in  $H$ , with  $\|f_n\| = 1$  for all  $n$ , there exists  $f \in H$  and a subsequence  $f_{n_k}$  such that for all  $g \in H$ :

$$\lim_{k \rightarrow \infty} (f_{n_k}, g) = (f, g).$$

One say that  $\{f_{n_k}\}$  converges weakly to  $f$ . **Hint:** you may run  $g$  through the basis of  $H$  and use the diagonalization argument and define  $f$  by its series expansion in the basis you used.

**Problem 4.** Suppose  $T$  is a bounded operator that is diagonal with respect to a basis  $\{\phi_k\}$ , with  $T\phi_k = \lambda_k\phi_k$ . Then  $T$  is a compact if and only if  $\lambda_k \rightarrow 0$ . **Hint:** Play with  $\|P_n T - T\|$  where  $P_n$  is the orthogonal projection on  $\phi_1, \phi_2, \dots, \phi_n$ .

**Problem 5.** Suppose  $H = L_2(B)$ , where  $B$  is the unit ball in  $\mathbb{R}^d$ . Let  $K(x, y)$  be a measurable function on  $B \times B$  that satisfies  $|K(x, y)| \leq A|x - y|^{-d+\alpha}$  for some  $\alpha > 0$ , whenever  $x, y \in B$ . Define

$$Tf(x) = \int_B K(x, y)f(y)dy.$$

- Prove that  $T$  is a bounded operator on  $H$ .
- Prove that  $T$  is compact. **Hint:** it may help to consider operators  $T_n$  with kernels  $K_n(x, y) = K(x, y)$  if  $|x - y| \geq 1/n$  and 0 otherwise. Show that each  $T_n$  is compact, and that  $\|T_n - T\| \rightarrow 0$  as  $n \rightarrow \infty$ .
- Note that  $T$  is a Hilbert-Schmidt operator iff  $\alpha > d/2$ .

**Problem 6.** Let  $H$  be a Hilbert space with basis  $\{\phi_k\}_{k=1}^\infty$ . Verify that the operator  $T$  defined as

$$T\phi_k = \frac{1}{k}\phi_{k+1},$$

is compact, but has no eigenvectors.