Functions of Real Variables 1 (62051/72051)
Home Work 2, due on Thursday SEPTEMBER 12.
Instructor: Prof. Artem Zvavitch.

Problem 1. *We may also define the outer measure by taking coverings by rectangles instead of cubes. More precisely, we define*

$$m_*^R(E) = \inf \sum_{j=1}^{\infty} |R_j|,$$

where the $\inf$ *is now taken over all countable coverings* $E \subset \bigcup_{j=1}^{\infty} R_j$ *by closed rectangles. Show that actually*

$$m_*^R(E) = m_*(E),$$

for every substet E *in* $\mathbb{R}^d$.

Problem 2. *Suppose* E *is a given set, and* O_n, *for* $n \in \mathbb{N}$, *is the set defined by*

$$O_n = \{x \in \mathbb{R}^d : d(x, E) < \frac{1}{n}\}.$$

- *Prove that* O_n *is open.*
- *Prove that if* E *is compact, then* $m(E) = \lim_{n\to\infty} m(O_n)$.
- *Would the above be true for* E *closed and unbounded set?*
- *Would the above be true for* E *open and bounded set?*

Problem 3. *Give an example of a set* A *such that the boundary* A *has positive Lebesque measure but* $m(A) = 0$.

Problem 4. *Consider a centered, unit euclidean ball* $B = \{x \in \mathbb{R}^d : |x| < 1\}$ *let* $v_d = m(B)$ *(note you are not required to know or to compute this constant!). Consider* $y \in \mathbb{R}^d$ *and* $r > 0$, *let* $B_r(y) = y + rB$ *(i.e. a euclidean ball of radius* r *centered at* y*). Prove that* $m(B_r(y)) = r^d v_d$.

Problem 5. *Let* Λ *be a diagonal matrix, with all* $\lambda_{i,i} > 0$. *Consider a measurable* $E \subset \mathbb{R}^d$. *Define*

$$\Lambda E = \{\Lambda x : x \in E\}.$$

Prove that ΛE *is measurable and* $m(\Lambda E) = [\det(\Lambda)] m(E)$.

Problem 6. *Let* A *be the subset of* $[0,1]$ *which consists of all numbers which do not have the digit* 4 *appearing in their decimal expansion. Find* $m(A)$.