Functions of Real Variables 2 (62052/72052)
Home WorkS 2 and 3, due on Monday, February 19.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Let $T : H_1 \to H_2$ be a linear operator, from finite dimensional Hilbert Space H_1 to a Hilbert space H_2. Prove that T is bounded.*

Problem 2. *Assume T is a bounded linear operator on a Hilbert space, prove that*

$$\|TT^*\| = \|T^*T\| = \|T\|^2 = \|T^*\|^2$$

Problem 3. *Assume H is an infinite-dimensional Hilbert space. Give an example of a sequence $\{f_n\}$ in H, with $\|f_n\| = 1$ for all n, such that $\{f_n\}$ has no converging subsequence. Next show that for any sequence $\{f_n\}$ in H, with $\|f_n\| = 1$ for all n, there exists $f \in H$ and a subsequence f_{n_k} such that for all $g \in H$:*

$$\lim_{k\to\infty}(f_{n_k}, g) = (f, g).$$

One say that $\{f_{n_k}\}$ converges weekly to f. **Hint:** *you may run g through the basis of H and use the diagonalization argument and define f by its series expansion in the basis you used.*

Problem 4. *Suppose T is a bounded operator that is diagonal with respect to a basis $\{\phi_k\}$, with $T\phi_k = \lambda_k \phi_k$. Then T is a compact if and only if $\lambda_k \to 0$.* **Hint:** *Play with $\|P_nT - T\|$ where P_n is the orthogonal projection on $\phi_1, \phi_2, \ldots, \phi_n$.*

Problem 5. *Suppose $H = L_2(B)$, where B is the unit ball in $\mathbb{R}^d$. Let $K(x, y)$ be a measurable function on $B \times B$ that satisfies $|K(x, y)| \le A|x - y|^{-d+\alpha}$ for some $\alpha > 0$, whenever $x, y \in B$. Define*

$$Tf(x) = \int_B K(x, y)f(y)dy.$$

- *Prove that T is a bounded operator on H.*
- *Prove that T is compact.* **Hint:** *it may help to consider operators T_n with kernels $K_n(x, y) = K(x, y)$ if $|x - y| \ge 1/n$ and 0 otherwise. Show that each T_n is compact, and that $\|T_n - T\| \to 0$ as $n \to \infty$.*
- *Prove that T is a Hilbert-Schmidt operator iff $\alpha > d/2$.*

Problem 6. *Let H be a Hilber space with basis $\{\phi_k\}_{k=1}^{\infty}$. Verify that the operator T defined as*

$$T\phi_k = \frac{1}{k}\phi_{k+1},$$

is compact, but has no eigenvectors.

Problem 7. *Consider the operator $T : L^2([0, 1]) \to L_2([0, 1])$ defined by*

$$T(f)(t) = tf(t).$$

Please, first prove that T is a bounded, symmetric linear operator, which is not compact. Next, show that T has no eigenvectors.

Problem 8. *Let H be a Hilbert space. Prove the following variants of the spectral theorem.*

- *T_1 and T_2 are two linear symmetric and compact operators on H that commute (i.e. $T_1T_2 = T_2T_1$), show that they can be diagonalized simultaneously. In other words, there exists an orthonorman basis for H which consists of eigenvectors for both T_1 and T_2.*

- *A linear operator on* H *is normal if* $TT^* = T^*T$. *Prove that if* T *is normal and compact, then* T *can be diagonalized.* **Hint:** *Write* $T = T_1 + iT_2$ *where* T_1 *and* T_2 *are symmetric, compact and commute.*
- *If* U *us unitary and* $U = \lambda I - T$ *where* T *is compact, then* U *can be diagonalized.*