Functions of Real Variables 2 (62051/72051)
Home Work 2, due on Wednesday , February 12.
Instructor: Prof. Artem Zvavitc./ Peter Gordon

Problem 1. *Let f be absolutely continuous in the interval $[\varepsilon, 1]$ for each $0 < \varepsilon < 1$. Does the continuity off f at 0 imply that f is absolutely continuous on $[0, 1]$? What if f is also of bounded variation on $[0, 1]$?*

Problem 2. *Prove the following variant of the* **Vitali covering lemma:** *If E is covered in the Vitali sense by a family $\mathcal{B}$ of balls, and $m_*(E) \in (0, \infty)$, then for every $\nu > 0$ there exists a disjoint collection of balls $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{B}$ such that*

$$m_*(E \setminus \bigcup_{j=1}^{\infty} B_j) = 0 \text{ and } \sum_{j=1}^{\infty} |B_j| \leq (1 + \nu) m_*(E).$$

Problem 3. *Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$.*

Problem 4. *Suppose F is an increasing function on $[a, b]$.*

- *Prove that we can decompose F as $F = F_A + F_C + F_J$, where F_A, F_C, F_J are increasing, F_A is absolutely continuous; F_C is continuous and F_J is a jump function.*
- *Please, also show that F_A, F_C and F_J are uniquely determined for each F up to an additive constant.*

Note, that the above construction is the Lebesgue decomposition of F.