Functions of Real Variables 1 (62051/72051)
Home Work 2, due on Monday, September 11.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Let $C \in [0,1]$ be Cantor set.*

- *Is it true that C contain irrational points?*
- *Is it true that C contains only boundary points (of C)?*
- *Please, find $\bar{C}$ (closure of C).*

Problem 2. *Consider the unit interval $[0,1]$, and let ξ be fixed real number with $\xi \in (0,1)$ (note that the case $\xi = 1/3$ corresponds to the regular Cantor set we learned in our lectures).*

In stage 1 of the construction, remove the centrally situated open interval in $[0,1]$ of length ξ. In stage 2 remove the centrally situated open intervals each of **relative** *length ξ (i.e. if the interval has length a you remove an interval of length $\xi \times a$), one in each of the remaining intervals after stage 1, and so on.*

Let C_ξ denote the set which remains after applying the above procedure indefinitely.

- *Prove that C_ξ is compact.*
- *Prove that C_ξ is totally disconnected and perfect.*
- *Atually, prove that the complement of C_ξ in $[0,1]$ is the union of open intervals of total length equal to 1.*
- *Show directly that $m_*(C_\xi) = 0$.*

Problem 3. *We may also define the outer measure by taking coverings by rectangles instead of cubes. More precisely, we define*

$$m_*^R(E) = \inf \sum_{j=1}^{\infty} |R_j|,$$

where the inf *is now taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} R_j$ by closed rectangles. Show that actually*

$$m_*^R(E) = m_*(E),$$

for every substet E in $\mathbb{R}^d$.

Problem 4. *Suppose E is a given set, and O_n, for $n \in \mathbb{N}$, is the set defined by*

$$O_n = \{x \in \mathbb{R}^d : d(x,E) < \frac{1}{n}\}.$$

- *Prove that O_n is open.*
- *Prove that if E is compact, then $m(E) = \lim_{n\to\infty} m(O_n)$.*
- *Would the above be true for E closed and unbounded set?*
- *Would the above be true for E open and bounded set?*