Functions of Real Variables 2 (62051/72051)
Home Work 1, due on Wednesday SEPTEMBER 14.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Suppose E is a given set, and O_n, for $n \in \mathbb{N}$, is the set defined by*

$$O_n = \{x \in \mathbb{R}^d : d(x, E) < \frac{1}{n}\}.$$

- *Prove that O_n is open.*
- *Prove that if E is compact, then $m(E) = \lim_{n\to\infty} m(O_n)$.*
- *Would the above be true for E closed and unbounded set?*
- *Would the above be true for E open and bounded set?*

Problem 2. *Give an example of an open set O such that the boundary of the closure of O has positive Lebesque measure. (***Hint:** *try to play with the union of open intervals taken out of $[0, 1]$ during the construction of Cantor set on...* **ODD** *steps.)*

Problem 3. *Consider a centered, unit euclidean ball $B = \{x \in \mathbb{R}^d : |x| < 1\}$ let $v_d = m(B)$ (note you are not required to know or to compute this constant!). Prove that $m(B_r(x)) = r^d v_d$.*

Problem 4. *Let Λ be a diagonal matrix, with all $\lambda_{i,i} > 0$. Consider a measurable $E \subset \mathbb{R}^d$. Define*

$$\Lambda E = \{\Lambda x : x \in E\}.$$

Prove that ΛE is measurable and $m(\Lambda E) = [\det(\Lambda)]m(E)$.

Problem 5. *Let A be the subset of $[0, 1]$ which consists of all numbers which do not have the diget 4 appearing in their decimal expansion. Find $m(A)$.*

Problem 6. *Show that a closed set is a G_δ and open set is F_δ.*