Functions of Real Variables 1 (62051/72051)
Home Work 3, due on Thursday SEPTEMBER 19.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Show that a closed set is a* G_δ *and open set is* F_σ.

Problem 2. *Suppose* $A \subset E \subset B$, *where* A, B *are measurable sets with* $m(A) = m(B)$. *Prove that* E *is measurable.*

Problem 3. *Let* $\mathcal{N}$ *denote the non-measurable subset of* $[0,1]$, *constructed in class and in the book.*

- *Prove that if* E *is a measurable subset of* $\mathcal{N}$, *then* $m(E) = 0$.
- *Assume that* G *is a subset of* $\mathbb{R}$ *with* $m_*(G) > 0$, *prove that there is a subset of* G *such that it is non-measurable.*
- *Prove that if* $\mathcal{N}^c = [0,1] \setminus \mathcal{N}$, *then* $m_*(\mathcal{N}^c) = 1$.
- *Now, conclude that*

$$m_*(\mathcal{N}) + m_*(\mathcal{N}^c) \neq m_*(\mathcal{N} \cup \mathcal{N}^c).$$

Problem 4. *Let* C_ξ *and* C_ν *be two Cantor sets (constructed in previous HW). Show that there exist a function* $F : [0,1] \to [0,1]$ *with the following properties*

- F *is continuous and bijective.*
- F *is monotonically increasing.*
- F *maps* C_ξ *surjectively onto* C_ν.
- *Now give an example of a measurable function* f *and a continuous function* Φ *so that* $f \circ \Phi$ *is non-measurable. One may use function* F *constructed above (BUT YOU NEED TO PLAY WITH IT). One of the ideas is to take two measurable sets* C_1 *and* C_2 *such that* $m(C_1) > 0$ *but* $m(C_2) = 0$ *and function* $\Phi : C_1 \to C_2$, *continuous. Also take* $\mathcal{N} \subset C_1$ *- non-measurable set and define* $f = \chi_{\Phi(\mathcal{N})}$.
- *Use the above construction to show that there exists a Lebesque measurable set that is not a Borel set.*

Problem 5. *Suppose that* Γ *is a curve* $(x, f(x))$ *in* $\mathbb{R}^2$, *where* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function. Prove that* $m(\Gamma) = 0$.