

Functions of Real Variables II (62052/72052)
Home Work 4, due on Wednesday, February 15.
Instructor: Prof. Artem Zvavitch.

Problem 1. Let S be a collection of subsets of \mathbb{R} that are countable or have countable complement. Prove that S is a σ -algebra.

Problem 2. Find all $p \in \mathbb{R}$ for which $d_p(x, y) = |x - y|^p$ gives a metric on \mathbb{R} .

Problem 3. Consider $d_p(x, y) = (\sum_{i=1}^n |x_i - y_i|^p)^{1/p}$, for $x, y \in \mathbb{R}^n$. Let $1 \leq p \leq q < \infty$, prove that

$$d_q(x, y) \leq d_p(x, y) \leq n^{\frac{1}{p} - \frac{1}{q}} d_q(x, y).$$

Use this remark to show that Borel σ -algebra on \mathbb{R}^n , defined with respect to $d_p(x, y)$ is independent of p .

Problem 4. For each $E \in \mathbb{R}^n$ let $m_*(E) = \inf \sum_{j=1}^n |Q_j|$, where the infimum is taken over all countable coverings of E by closed cubes. Prove that $m_*(E)$ is an exterior measure.

Problem 5. We say that $E \in \mathbb{R}^n$ is Lebesgue measurable, if for any $\varepsilon > 0$, there exists open set $O \in \mathbb{R}^n$ such that $E \subset O$ and $m_*(O - E) \leq \varepsilon$.

Prove that the $E \subset \mathbb{R}^n$ is Lebesgue measurable if and only if it is Caratheodory measurable.