

**Functions of Real Variables 1 (62051/72051)**  
**Home Work 4, due on Thursday October 1.**  
**Instructor: Prof. Artem Zvavitch.**

**Problem 1. (The Borel-Cantelli lemma):** Suppose  $\{E_k\}_{k=1}^{\infty}$  is countable family of measurable subsets of  $\mathbb{R}^d$  and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\} = \limsup_{k \rightarrow \infty} (E_k).$$

- Show that  $E$  is measurable.
- Prove that  $m(E) = 0$ .

**Hint:** write  $E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$ .

**Problem 2.** Let  $\{f_n\}$  be a sequence of measurable functions on  $[0, 1]$  with  $|f_n(x)| < \infty$  for a.e.  $x$ . Show that there exists a sequence  $c_n$  of positive real numbers such that

$$\frac{f_n(x)}{c_n} \rightarrow 0, \text{ a.e. } x.$$

**Hint:** Pick  $c_n$  such that  $m(\{x : |f_n(x)/c_n| > 1/n\}) < 2^{-n}$ , and apply the Borel-Cantelli lemma.

**Problem 3.** Prove that there is a continuous function that maps a Lebesgue measurable set to non-measurable set. Discussion around Problem 4 in HW 3 may help.

**Problem 4.** Show that there is no function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

- $f$  is continuous.
- $f = \chi_{[0,1]}$  a.e. on  $\mathbb{R}$ .