Functions of Real Variables II (62052/72052)
Home Work 4, due on Tuesday, March 3.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Let X be a set and M be a non-empty collection of subsets of X. Prove that if M is closed under complements and countable unions of disjoint sets, then M is a σ-algebra.*

Problem 2. *Let S be a collection of subsets of $\mathbb{R}$ that are countable or have countable complement. Prove that S is a σ-algebra.*

Problem 3. *Find all $p \in \mathbb{R}$ for which $d_p(x,y) = |x-y|^p$ gives a metric on $\mathbb{R}$.*

Problem 4. *Consider $d_p(x,y) = (\sum_{i=1}^n |x_i - y_i|^p)^{1/p}$, for $x, y \in \mathbb{R}^n$. Let $1 \le p \le q < \infty$, prove that*

$$d_q(x,y) \le d_p(x,y) \le n^{\frac{1}{p}-\frac{1}{q}} d_q(x,y).$$

Use this remark to show that Borel σ-algebra on $\mathbb{R}^n$, defined with respect to $d_p(x,y)$ is independent of p.

Problem 5. *For each $E \in \mathbb{R}^n$ let*

$$m_*(E) = \inf \sum_{j=1}^{n} |Q_j|,$$

where the infimum is taken over all countable coverings of E by closed cubes. Prove that $m_*(E)$ is an exterior measure.

Problem 6. *We say that $E \in \mathbb{R}^n$ is Lebesque measurable, if for any $\varepsilon > 0$, there exists open set $O \subset \mathbb{R}^n$ such that $E \subset O$ and $m_*(O - E) \le \varepsilon$. Please, prove that the $E \subset \mathbb{R}^n$ is Lebesque measurable if and only if it is Caratheodory measurable.*