

Functions of Real Variables 1 (62051/72051)
Home Work 4, due on Wednesday SEPTEMBER 27.
Instructor: Prof. Artem Zvavitch.

Problem 1. Let \mathcal{N} denote the non-measurable subset of $[0, 1]$, constructed in class and in the book.

- Prove that if E is a measurable subset of \mathcal{N} , then $m(E) = 0$.
- Assume that G is a subset of \mathbb{R} with $m_*(G) > 0$, prove that there is a subset of G such that it is non-measurable.
- Prove that if $\mathcal{N}^c = [0, 1] \setminus \mathcal{N}$, then $m_*(\mathcal{N}^c) = 1$.
- Now, conclude that

$$m_*(\mathcal{N}) + m_*(\mathcal{N}^c) \neq m_*(\mathcal{N} \cup \mathcal{N}^c).$$

Problem 2. Let C_ξ and C_ν be two Cantor sets (constructed in previous HW). Show that there exist a function $F : [0, 1] \rightarrow [0, 1]$ with the following properties

- F is continuous and bijective.
- F is monotonically increasing.
- F maps C_ξ surjectively onto C_ν .
- Now give an example of a measurable function f and a continuous function Φ so that $f \circ \Phi$ is non-measurable. One may use function F constructed above (BUT YOU NEED TO PLAY WITH IT). One of the ideas is to take two measurable sets C_1 and C_2 such that $m(C_1) > 0$ but $m(C_2) = 0$ and function $\Phi : C_1 \rightarrow C_2$, continuous. Also take $\mathcal{N} \subset C_1$ - non-measurable set and define $f = \chi_{\Phi(\mathcal{N})}$.
- Use the above construction to show that there exists a Lebesgue measurable set that is not a Borel set.

Problem 3. Prove that there is a continuous function that maps a Lebesgue measurable set to non-measurable set. The previous problem may help!

Problem 4. Suppose that Γ is a curve $(x, f(x))$ in \mathbb{R}^2 , where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Prove that $m(\Gamma) = 0$.

Problem 5. (The Borel-Cantelli lemma): Suppose $\{E_k\}_{k=1}^\infty$ is countable family of measurable subsets of \mathbb{R}^d and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\} = \limsup_{k \rightarrow \infty} (E_k).$$

- Show that E is measurable.
- Prove that $m(E) = 0$.

Hint: write $E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$.

Problem 6. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ with $|f_n(x)| < \infty$ for a.e. x . Show that there exists a sequence c_n of positive real numbers such that

$$\frac{f_n(x)}{c_n} \rightarrow 0, \text{ a.e. } x.$$

Hint: Pick c_n such that $m(\{x : |f_n(x)/c_n| > 1/n\}) < 2^{-n}$, and apply the Borel-Cantelli lemma.