

Functions of Real Variables 2 (62052/72052)
Home WorkS 4 and 5, due on Wednesday, April 2.
Instructor: Prof. Artem Zvavitch.

Problem 1. Let $T : H_1 \rightarrow H_2$ be a linear operator, from finite dimensional Hilbert Space H_1 to a Hilbert space H_2 . Prove that T is bounded.

Problem 2. Assume T is a bounded linear operator on a Hilbert space, prove that

$$\|TT^*\| = \|T^*T\| = \|T\|^2 = \|T^*\|^2$$

Problem 3. Assume H is an infinite-dimensional Hilbert space. Give an example of a sequence $\{f_n\}$ in H , with $\|f_n\| = 1$ for all n , such that $\{f_n\}$ has no converging subsequence. Next show that for any sequence $\{f_n\}$ in H , with $\|f_n\| = 1$ for all n , there exists $f \in H$ and a subsequence f_{n_k} such that for all $g \in H$:

$$\lim_{k \rightarrow \infty} (f_{n_k}, g) = (f, g).$$

One say that $\{f_{n_k}\}$ converges weekly to f . **Hint:** you may run g through the basis of H and use the diagonalization argument and define f by its series expansion in the basis you used.

Problem 4. Suppose T is a bounded operator that is diagonal with respect to a basis $\{\phi_k\}$, with $T\phi_k = \lambda_k\phi_k$. Then T is a compact if and only if $\lambda_k \rightarrow 0$. **Hint:** Play with $\|P_n T - T\|$ where P_n is the orthogonal projection on $\phi_1, \phi_2, \dots, \phi_n$.

Problem 5. Suppose $H = L_2(B)$, where B is the unit ball in \mathbb{R}^d . Let $K(x, y)$ be a measurable function on $B \times B$ that satisfies $|K(x, y)| \leq A|x - y|^{-d+\alpha}$ for some $\alpha > 0$, whenever $x, y \in B$. Define

$$Tf(x) = \int_B K(x, y)f(y)dy.$$

- Prove that T is a bounded operator on H .
- Prove that T is compact. **Hint:** it may help to consider operators T_n with kernels $K_n(x, y) = K(x, y)$ if $|x - y| \geq 1/n$ and 0 otherwise. Show that each T_n is compact, and that $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$.
- Prove that T is a Hilbert-Schmidt operator iff $\alpha > d/2$.

Problem 6. Let H be a Hilbert space with basis $\{\phi_k\}_{k=1}^\infty$. Verify that the operator T defined as

$$T\phi_k = \frac{1}{k}\phi_{k+1},$$

is compact, but has no eigenvectors.

Problem 7. Consider the operator $T : L^2([0, 1]) \rightarrow L^2([0, 1])$ defined by

$$T(f)(t) = tf(t).$$

Please, first prove that T is a bounded, symmetric linear operator, which is not compact. Next, show that T has no eigenvectors.

Problem 8. Let H be a Hilbert space. Prove the following variants of the spectral theorem.

- T_1 and T_2 are two linear symmetric and compact operators on H that commute (i.e. $T_1T_2 = T_2T_1$), show that they can be diagonalized simultaneously. In other words, there exists an orthonormal basis for H which consists of eigenvectors for both T_1 and T_2 .

- A linear operator on H is normal if $TT^* = T^*T$. Prove that if T is normal and compact, then T can be diagonalized. **Hint:** Write $T = T_1 + iT_2$ where T_1 and T_2 are symmetric, compact and commute.
- If U is unitary and $U = \lambda I - T$ where T is compact, then U can be diagonalized.