Functions of Real Variables II (62052/72052)
Home Work 5, due on Friday, March 3.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Let $\mathcal{A}$ be an algebra of subsets of X, μ_0 be a pre-measure defined on $\mathcal{A}$, μ_* be the exterior measure generated by μ_0. Let $\mathcal{A}_\sigma$ be the collection of sets that are countable unions of sets in $\mathcal{A}$ and $\mathcal{A}_{\sigma\delta}$ the sets that arise as countable intersection of sets in $\mathcal{A}_\sigma$. Please prove that for any $B \subset X$ and any $\varepsilon > 0$, there are sets $E_1 \in \mathcal{A}_\sigma$ and $E_2 \in \mathcal{A}_{\sigma\delta}$, such that $B \subset E_1$, $B \subset E_2$ and*

$$\mu_*(E_1) \leq \mu_*(B) + \varepsilon \text{ and } \mu_*(B) = \mu_*(E_2).$$

Problem 2. *Let $\{(X_\alpha, \mathcal{M}_\alpha, \mu_\alpha)\}$ be a collection of measure spaces, and suppose hat the sets $\{X_\alpha\}$ are all disjoint. Then we can form a new measure space (called their union) by letting*

$$X = \cup X_\alpha, \quad \mathcal{M} = \{B : B \cap X_\alpha \in \mathcal{M}_\alpha, \text{ for all } \alpha\}$$

define $\mu(B) = \sum \mu_\alpha(B \cap X_\alpha)$. Please show that

- *$\mathcal{M}$ is a σ - algebra.*
- *μ is a measure.*
- *Shoe that μ is σ-finite iff all but a countable number of μ_α are zero and the remainder are σ-finite.*

Problem 3. *Consider a measure space $(X, \mathcal{M}, \mu)$,*

- *Consider $A, B \in \mathcal{M}$ such that $\mu(A \triangle B) = 0$, show that $\mu(A) = \mu(B)$.*
- *We say that measure μ is complete, if from $\mu(F) = 0$ we get that $E \in \mathcal{M}$ for every $E \subset F$. Assume that μ is a complete measure. Prove that if $A \subset X$ such that there exists $B \in \mathcal{M}$ such that $\mu(A \triangle B) = 0$, then $A \in \mathcal{M}$.*
- *Fix set $Y \in \mathcal{M}$ let $\mathcal{M}_Y = \{A : A \in \mathcal{M} \text{ and } A \subset Y\}$ and $\mu_Y(A) = \mu(A)$, for $A \in \mathcal{M}_Y$. Prove that $(Y, \mathcal{M}_Y, \mu_Y)$ is a measure space. The measure μ_Y is called* **restriction** *of μ to Y.*
- *Consider another measure ν on $\mathcal{M}$. Prove that $\lambda(B) = \mu(B) + \nu(B)$, for $B \in \mathcal{M}$ is a measure.*
- *Prove that if λ is a measure on $\mathcal{M}$ such that $\lambda(B) \geq \mu(B)$, for all $B \in \mathcal{M}$, then there exists a measure ν on $\mathcal{M}$, such that $\lambda = \mu + \nu$.*
- *Prove that if ν is σ - finite, then the measure ν from above is unique.*