

**Functions of Real Variables 1 (62051/72051)**  
**Home Work 5, due on Thursday October 15.**  
**Instructor: Prof. Artem Zvavitch.**

**Problem 1.** Prove that if  $f$  is integrable on  $\mathbb{R}^d$  and  $\delta > 0$ , then  $f(\delta x)$  converges to  $f(x)$  in the  $L^1$ -norm as  $\delta \rightarrow 1$ .

**Problem 2.** Suppose  $f$  is integrable on  $[0, b]$ , and

$$g(x) = \int_x^b \frac{f(t)}{t} dt, \text{ for } x \in (0, b].$$

Prove that  $g$  is integrable on  $[0, b]$  and

$$\int_0^b g(x) dx = \int_0^b f(t) dt.$$

**Problem 3.** If  $f$  is integrable on  $\mathbb{R}$  show that  $F(x) = \int_{-\infty}^x f(t) dt$  is uniformly continuous.

**Problem 4. (Markov's inequality:)** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f(x) > 0$  and  $f$  is integrable. Prove that

$$m(\{x : f(x) > t\}) \leq \frac{1}{t} \int f(x) dx,$$

for all  $t > 0$ .

**Problem 5. (Chebyshev's inequality:)** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , and  $|f|^p$  is integrable, for some  $p > 0$ . Prove that

$$m(\{x : f(x) > t\}) \leq \frac{1}{t^p} \int |f(x)|^p dx,$$

for all  $t > 0$ .

**Problem 6.** Suppose  $f \geq 0$ , let

$$E_{2^k} = \{x : f(x) > 2^k\} \text{ and } F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}.$$

If  $f$  is finite, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{x : f(x) > 0\}$$

and the sets  $F_k$  are disjoint.

Prove that  $f$  is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \text{ and if and only if } \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

Please, use this result to verify the following assertions: let

$$f(x) = \begin{cases} |x|^{-a}, & \text{if } |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases} \text{ and } g(x) = \begin{cases} |x|^{-b}, & \text{if } |x| > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Where  $|x|$  is the length of  $x \in \mathbb{R}^d$ . Then  $f$  is integrable on  $\mathbb{R}^d$  if and only if  $a < d$ ; also  $g$  is integrable on  $\mathbb{R}^d$  if and only if  $b > d$ .