

Functions of Real Variables II (62052/72052)

Home Work 5, due on Friday, March 10.

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Problem 1. Let \mathcal{A} be an algebra of subsets of X , μ_0 be a pre-measure defined on \mathcal{A} , μ_* be the exterior measure generated by μ_0 . Let \mathcal{A}_σ be the collection of sets that are countable unions of sets in \mathcal{A} and $\mathcal{A}_{\sigma\delta}$ the sets that arise as countable intersection of sets in \mathcal{A}_σ . Please prove that for any $B \subset X$ and any $\varepsilon > 0$, there are sets $E_1 \in \mathcal{A}_\sigma$ and $E_2 \in \mathcal{A}_{\sigma\delta}$, such that $B \subset E_1$, $B \subset E_2$ and

$$\mu_*(E_1) \leq \mu_*(B) + \varepsilon \text{ and } \mu_*(B) = \mu_*(E_2).$$

Problem 2. Let $\{(X_\alpha, \mathcal{M}_\alpha, \mu_\alpha)\}$ be a collection of measure spaces, and suppose that the sets $\{X_\alpha\}$ are all disjoint. Then we can form a new measure space (called their union) by letting

$$X = \cup X_\alpha, \quad \mathcal{M} = \{B : B \cap X_\alpha \in \mathcal{M}_\alpha, \text{ for all } \alpha\}$$

define $\mu(B) = \sum \mu_\alpha(B \cap X_\alpha)$. Please show that

- \mathcal{M} is a σ - algebra.
- μ is a measure.
- Show that μ is σ -finite iff all but a countable number of μ_α are zero and the remainder are σ -finite.

Problem 3. Consider a measure space (X, \mathcal{M}, μ) ,

- Consider $A, B \in \mathcal{M}$ such that $\mu(A \triangle B) = 0$, show that $\mu(A) = \mu(B)$.
- We say that measure μ is complete, if from $\mu(F) = 0$ we get that $E \in \mathcal{M}$ for every $E \subset F$. Assume that μ is a complete measure. Prove that if $A \subset X$ such that there exists $B \in \mathcal{M}$ such that $\mu(A \triangle B) = 0$, then $A \in \mathcal{M}$.
- Fix set $Y \in \mathcal{M}$ let $\mathcal{M}_Y = \{A : A \in \mathcal{M} \text{ and } A \subset Y\}$ and $\mu_Y(A) = \mu(A)$, for $A \in \mathcal{M}_Y$. Prove that $(Y, \mathcal{M}_Y, \mu_Y)$ is a measure space. The measure μ_Y is called **restriction** of μ to Y .
- Consider another measure ν on \mathcal{M} . Prove that $\lambda(B) = \mu(B) + \nu(B)$, for $B \in \mathcal{M}$ is a measure.
- Prove that if λ is a measure on \mathcal{M} such that $\lambda(B) \geq \mu(B)$, for all $B \in \mathcal{M}$, then there exists a measure ν on \mathcal{M} , such that $\lambda = \mu + \nu$.
- Prove that if λ is σ - finite, then the measure ν from above is unique.