Functions of Real Variables II (62052/72052)
Home Work 6, due on Tuesday, March 31.
Instructor: Prof. Artem Zvavitch.

In this HW, we will talk about Lebesgue measure. The first two problems belong to the previous semester and were (at least partially) discussed there.

Problem 1. *Suppose L is a linear transformation of $\mathbb{R}^n$. Show that if E is (Lebesgue) measurable subset of $\mathbb{R}^n$, then $L(E)$ is also measurable. You may follow the following steps:*

- *Note that if E is compact, so is $L(E)$. Hence if E is $\mathbf{F}_\sigma$ set (countable union of closed sets), so is $L(E)$.*
- *Show that L will satisfy*

$$|L(x) - L(x')| \leq M|x - x'|$$

for some $M > 0$ and all $x, x' \in \mathbb{R}^n$. Thus L maps any cube of side length l into a cube of side length $2\sqrt{n}Ml$. Now if $m(E) = 0$, there is a collection of cubes $\{Q_j\}$ such that $E \subset \cup Q_j$, and $\sum m(Q_j) < \varepsilon$. Thus $m_*(L(E)) \leq c'\varepsilon$ and hence $m(L(E)) = 0$. Combine this with the property that E is measurable if differs from a $\mathbf{F}_\sigma$ set by a measure zero.

Problem 2. *Suppose L is a linear transformation of $\mathbb{R}^n$. Show that if E is (Lebesque) measurable subset of $\mathbb{R}^n$, then $m(L(E)) = |\det L| m(E)$*

For the next few problems, let $\Phi = (\Phi_1(x), \Phi_2(x), \dots, \Phi_n(x))$ be a bijection of open set $O \subset \mathbb{R}^n$ onto another open set $O' \subset \mathbb{R}^n$, such that Φ is of the class C^1 (i.e. each $\frac{\partial \Phi_k}{\partial x_i}$, $k, i \in 1, 2, \dots, n$ exist and continuous on O).

Problem 3. *Follow ideas proposes for a solution of Problem 1 to show that for every measurable $E \subset O$ we get $\Phi(E)$ is measurable.*

Problem 4. (See hint on the next page) *Prove that if $Q \subset O$ is a cube with diameter ε (i.e. the length of the diagonal of the cube is less then ϵ) and z is a center of Q then*

$$\Phi(x) = \Phi(z) + \Phi'(z)(x - z) + o(\varepsilon)$$

for all $x \in Q$. Note that, using this property we get

$$\Phi(Q) = \Phi(z) + \Phi'(z)(Q - z) + o(\varepsilon).$$

Problem 5. *Prove that $m(\Phi(E)) = \int_E |\det \Phi'(x)| dx$, where Φ' is the Jacobian of Φ. Here you may assume that E is a bounded open set, and write $E = \cup Q_k$, where Q_k are cubes whose interiors are disjoint, and whose diameter is less than ε. Let z_k be the center of each Q_k. Then use the previous problem to show that there exists function η such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

$$(1 - \eta(\varepsilon))\Phi'(z_k)(Q_k - z_k) \subset \Phi(Q_k) - \Phi(z_k) \subset (1 + \eta(\varepsilon))\Phi'(z_k)(Q_k - z_k).$$

This would give you

$$m(\Phi(O)) = \sum m(\Phi(Q_k)) = \sum |\det(\Phi'(z_k))| m(Q_k) + o(1), \text{ as } \varepsilon \to 0,$$

where you, need to use Problem 2 for the second equality.

Problem 6. *Prove that*

$$\int_{O'} f(y)dy = \int_O f(\Phi(x))|\det \Phi'(x)|dx,$$

for every integrable function f on O'.

Hint for Problem 4: First of all let me remind the notation $\Phi'(z)$ is a matrix (Jacobian) with elements $\frac{\partial \Phi_k(z)}{\partial x_i}$.

We need to use Taylor's theorem: assume f is differentiable on $\mathbb{R}$ and $a, b \in \mathbb{R}$, then

$$f(a) = f(b) + f'(b)(a-b) + o(|a-b|),$$

here $o(t)$ is such that $o(t)/t \to 0$ as $t \to 0$. Check the books for different forms of the Reminder. Now we need to create a multidimensional version of it. Consider our map $\Phi(x) = (\Phi_1(x), \Phi_2(x), \dots, \Phi_n(x))$. We will work just with one of the functions $\Phi_k(x)$, our goal to estimate the difference of $\Phi_k(x)$ and $\Phi_k(z)$, where x, z are as in Problem 4. Consider function

$$y(t) = tx + (1-t)z, \text{ where } t \in [0,1].$$

Note $y(0) = z$ and $y(1) = x$. Now consider function $f(t) = \Phi_k(y(t))$, apply Taylor's formula

$$f(1) = f(0) + f'(0)(1-0) + o(|x-z|),$$

Now you check that

$$f(1) = \Phi_k(x) \text{ and } f(0) = \Phi_k(z)$$

$$f'(0) = \frac{d}{dt}\Phi_k(y(t))|_{t=0} = \sum_{i=1}^{n} \frac{\partial \Phi_k(y(t))}{\partial x_i}(x_i - z_i)|_{t=0} = \sum_{i=1}^{n} \frac{\partial \Phi_k(z)}{\partial x_i}(x_i - z_i).$$