Functions of Real Variables 1 (62051/72051)
Home Work 6, due on Wednesday October 18.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Suppose f is integrable on $[0,b]$, and*

$$g(x) = \int_x^b \frac{f(t)}{t} dt, \text{ for } x \in (0,b].$$

Prove that g is integrable on $[0,b]$ and

$$\int_0^b g(x)dx = \int_0^b f(t)dt.$$

Problem 2. *If f is integrable on $\mathbb{R}$ show that $F(x) = \int_{-\infty}^x f(t)dt$ is uniformly continuous.*

Problem 3. (Markov's inequality:) *Suppose $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) > 0$ and f is integrable. Prove that*

$$m(\{x : f(x) > t\}) \leq \frac{1}{t} \int f(x)dx,$$

for all $t > 0$.

Problem 4. (Chebyshev's inequality:) *Suppose $f : \mathbb{R}^n \to \mathbb{R}$, and $|f|^p$ is integrable, for some $p > 0$. Prove that*

$$m(\{x : f(x) > t\}) \leq \frac{1}{t^p} \int |f(x)|^p dx,$$

for all $t > 0$.

Problem 5. *Suppose $f \geq 0$, let*

$$E_{2^k} = \{x : f(x) > 2^k\} \text{ and } F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}.$$

If f is finite, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{x : f(x) > 0\}$$

and the sets F_k are disjoint.

Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \text{ and if and only if } \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

Please, use this result to verify the following assertions: let

$$f(x) = \begin{cases} |x|^{-a}, & \text{if } |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases} \text{ and } g(x) = \begin{cases} |x|^{-b}, & \text{if } |x| > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Where $|x|$ is the length of $x \in \mathbb{R}^d$. Then f is integrable on $\mathbb{R}^d$ if and only if $a < d$; also g is integrable on $\mathbb{R}^d$ if and only if $b > d$.