

Functions of Real Variables 1 (62051/72051)
Home Work 7, due on Tuesday November 5.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Let's study the integrability properties of f^* :*

- *Prove that if f is integrable on \mathbb{R}^d , and f is not identically zero, then*

$$f^*(x) \geq \frac{c}{|x|^d}, \text{ for some } c > 0 \text{ and all } |x| \geq 1.$$

- *Now, please, Conclude that f^* is not integrable on \mathbb{R}^d .*
- *Show that the weak type estimate*

$$m(\{x : f^*(x) > \alpha\}) \leq \frac{c}{\alpha}$$

for all $\alpha > 0$, whenever $\int |f| = 1$ is best possible in the following sense: if f is supported in the unit ball with $\int |f| = 1$, then

$$m(\{x : f^*(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

for some $c' > 0$ and all sufficiently small α .

I would suggest to start with an observation that if f is not identically zero then there always exists a ball B such that $\int_B |f| > 0$.

Problem 2. *Consider the function on \mathbb{R} defined by*

$$f(x) = \begin{cases} \frac{1}{|x|(\log 1/|x|)^2}, & \text{if } |x| \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

- *Show that f is integrable.*
- *Establish the inequality*

$$f^*(x) \geq \frac{c}{|x|(\log 1/|x|)} \text{ for some } c > 0 \text{ and } |x| \leq 1/2,$$

to conclude that the maximal function f^ is not locally integrable.*

Problem 3. *Prove that if a measurable subset $E \subset [0, 1]$ satisfies $m(E \cap I) \geq \alpha m(I)$, for some $\alpha > 0$ and all intervals $I \subset [0, 1]$, then $m(E) = 1$. **Hint:** A corollary about points of density proved in the class, may help.*