Functions of Real Variables II (62052/72052)
Home Work 7, due on Tuesday, April 7.
Instructor: Prof. Artem Zvavitch.

Problem 1. *The goal of this problem is to prove that the Lebesgue measure is uniquely characterized by its translation invariance. More precesely: Let μ be a Borel measure on $\mathbb{R}^n$ that is translation-invariant, and is finite on compact sets, then μ is a multiple of Lebesgue measure m. Please, prove this fact by following those steps:*

- *Suppose Q_a denotes a translation of the cube $(0,a]^d$. If we let $\mu Q_1 = c$, then $\mu(Q_{1/n}) = cn^{-d}$ for each natural number n.*
- *As a result μ is absolutely continuous with respect to m, and there is a locally integrable function f such that*
$$\mu(E) = \int_E f dx.$$
- *Now you may apply the differentiation theorem (check a theorem in previous semester that was talking about sets U_α which shrinks regularly to x and the limit of $\frac{1}{m(U_\alpha)}\int_{U_\alpha} f(y)dy$) to show that $f(x) = c$ a.e., and hence $\mu = cm$.*

Problem 2. *Let $C([a,b])$ be a space of continuous function on the closed, bounded interval $[a,b]$. Let μ be a Borel measure, such that $\mu([a,b]) < \infty$. Then*
$$f \to \ell(f) = \int_a^b f(x)d\mu(x)$$
is a linear functional on $C([a,b])$, with ℓ positive in the sense that $\ell(f) \geq 0$ if $f \geq 0$.

Please, prove that, conversely, for any linear functional ℓ on $C([a,b])$ that is positive, there is a unique Borel measure μ so that $\ell(f) = \int_a^b f(x)d\mu(x)$ for all $f \in C([a,b])$.

Hint: *The idea is to use the theorem on increasing, continuous from the right functions we proved in class to study Lebesgue - Stieltjes integral. Let $[a,b] = [0,1]$. We take $F(u) = \lim_{\varepsilon\to 0} \ell(f_\varepsilon)$, where $f_\varepsilon(x) = 1$, if $x \in [0,u]$, $f_\varepsilon(x) = 0$, if $x \in [u+\varepsilon, 1]$ and connected by a linear segment for $x \in [u, u+\varepsilon]$. Then F is increasing and right-continuous, moreover, $\ell(f)$ can be written as $\int_a^b f d\mu$.*

Problem 3. *Suppose ν_1, ν_2 and ν_3 are signed measures on $(X, \mathcal{M})$ and μ a (positive) measure on $\mathcal{M}$. Please, prove*

- *If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ then $(\nu_1 + \nu_2) \perp \mu$.*
- *If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.*
- *$\nu_1 \perp \nu_2$ implies $|\nu_1| \perp |\nu_2|$.*
- *$\nu \ll |\nu|$, for each signed measure ν.*
- *If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.*

Is it true that if

- *$\nu_1 \ll \nu_2$ and $\nu_2 \ll \nu_3$ then $\nu_1 \ll \nu_3$?*
- *$\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ then $\nu_1 \ll \nu_2$?*
- *If $\nu_1 \ll \nu_2$ then $\nu_2 \ll \nu_1$?*