Functions of Real Variables 1 (62051/72051)
Home Work 07, due on Wednesday November 1.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Suppose f is integrable on $[0, b]$, and*

$$g(x) = \int_x^b \frac{f(t)}{t} dt, \text{ for } x \in (0, b].$$

Prove that g is integrable on $[0, b]$ and

$$\int_0^b g(x)dx = \int_0^b f(t)dt.$$

Problem 2. *Give an example of two measurable sets A and B such that $A+B$ is not measurable (**Hint:** we WILL do it in class after discussion of Fubini's theorem.).*

Problem 3. Volume of the Unit Ball: *We proved before that if $B = \{x \in \mathbb{R}^d : |x| \le 1\}$, then $m(rB) = r^d m(B)$ for all $r \ge 0$. Now it is the time to provide the formula for $v_d = m(B)$:*

- *Use Fubini's theorem (Corollary of Fubini's theorem) to shoe that*

$$v_2 = 2 \int_{-1}^{1} (1 - x^2)^{1/2} dx,$$

  and show (using standard tricks for Riemann integral) that $v_2 = \pi$.
- *Use similar methods to show that*

$$v_d = 2v_{d-1} \int_0^1 (1 - x^2)^{(d-1)/2} dx.$$

- *Show that*

$$v_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}.$$

  Where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$.
- *Prove (use induction) that $Gamma(n) = (n-1)!$ and write a formula for v_{2k+1}.*

Problem 4. *Suppose $f \in L_1(\mathbb{R}^d)$. For each $t > 0$ let $E_t = \{x : |f(x)| > t\}$. Prove that*

$$\int_{\mathbb{R}^d} |f(x)|dx = \int_0^\infty m(E_t)dt.$$

Problem 5. Convolution of two functions. *Consider two measurable functions $f, g : \mathbb{R}^d \to \mathbb{R}$.*

- *Prove that $f(x - y)g(y)$ is measurable of $\mathbb{R}^{2d}$.*
- *Show that if $f, g \in L_1(\mathbb{R}^d)$, then $f(x - y)g(y) \in L_1(\mathbb{R}^{2d})$.*
- *Now we can define a convolution operator, which takes two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ and gives function $f * g : \mathbb{R}^d \to R$ defined as*

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy.$$

  Show that $f * g$ is well defined for a.e. x (i.e. $f(x - y)g(y)$ is integrable for a.e. x).

- *Show that* $f * g$ *is integrable whenever* f *and* g *are integrable, and that*

$$\|f * g\|_{L_1(\mathbb{R}^d)} \leq \|f\|_{L_1(\mathbb{R}^d)} \|g\|_{L_1(\mathbb{R}^d)}.$$

- *Show that* $f * g$ *is uniformly continuous when* f *is integrable and* g *bounded.*
- *Show that if in addition to above* g *is integrable, then*

$$(f * g)(x) \to 0 \text{ as } |x| \to \infty.$$