

Functions of Real Variables 1 (62051/72051)
Home Work 8, due on Tuesday November 12.
Instructor: Prof. Artem Zvavitch.

Problem 1. Suppose $\{K_\delta\}$ is a family of kernels on \mathbb{R}^d that satisfy

- (1) $|K_\delta(x)| \leq A\delta^{-d}$ for all $\delta > 0$,
- (2) $|K_\delta(x)| \leq A\delta/|x|^{d+1}$ for all $\delta > 0$,
- (3) $\int_{\mathbb{R}^d} K_\delta(x)dx = 0$ for all $\delta > 0$.

Show that if $f \in L_1(\mathbb{R}^d)$, then

$$(f * K_\delta)(x) \rightarrow 0 \text{ for a.e. } x, \text{ as } \delta \rightarrow 0.$$

Problem 2. Assume that F is continuous on $[a, b]$. Show that

$$D^+(F)(x) = \limsup_{h \rightarrow 0^+} \frac{F(x+h) - F(x)}{h}$$

is measurable.

Problem 3. Assume F is of bounded variation and continuous. Prove that $F = F_1 - F_2$, where both F_1 and F_2 are monotonic and continuous.

Problem 4. Show that if F is of bounded variation in $[a, b]$, then

- $\int_a^b |F'(x)|dx \leq T_F(a, b)$.
- $\int_a^b |F'(x)|dx = T_F(a, b)$ iff F is absolutely continuous.

Use the last result to show that the formula $L = \int_a^b |z'(t)|dt$ for the length of a rectifiable curve parametrized by $z(t)$ holds iff z is absolutely continuous.

Problem 5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Prove that f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq M|x - y|,$$

for some M and all $x, y \in \mathbb{R}$, iff f satisfies the following two properties:

- f is absolutely continuous.
- $|f'(x)| \leq M$.