Functions of Real Variables II (62052/72052)
Home Work 8, due on Thursday, April 16.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Show that the mean ergodic theorem still holds if we replace the assumption that T is an isometry by the assumption that T is contraction, that is $\|Tf\| \leq \|f\|$ for all $f \in \mathcal{H}$.* **Hint:** *Prove that T is contraction iff T^* is a contraction (you may use some discussions in the lecture notes as well as identity that $(f, T^*f) = (Tf, f)$).*

Problem 2. *Let τ be a measure-preserving transformation on (X, μ) with $\mu(X) = 1$. Prove that then τ is ergodic if and only if whenever ν is absolutely continuous with respect to μ and ν is τ-invariant (i.e. $\nu(\tau^{-1}(E)) = \nu(E)$ for all measurable sets E), then $\nu = c\mu$ with c a constant.*

Problem 3. *Suppose τ is a measure-preserving transformation on (X, μ). If*

$$\mu(\tau^{-n}(E) \cap F) \to \mu(E)\mu(F)$$

*as $n \to \infty$ for all measurable sets E and F, then $(T^n f, g) \to (f, 1)(g, 1)$, whenever $f, g \in L^2(X, \mu)$ with $(Tf(x)) = f(\tau(x))$. Thus τ is mixing. (***Hint:** *you may start with simple functions).*