Functions of Real Variables 1 (62051/72051)
Home Work 9, due on Tuesday November 26.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Prove the following variant of the* **Vitali covering lemma:** *If E is covered in the Vitali sense by a family $\mathcal{B}$ of balls, and $m_*(E) \in (0,\infty)$, then for every $\nu > 0$ there exists a disjoint collection of balls $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{B}$ such that*

$$m_*(E \setminus \bigcup_{j=1}^{\infty} B_j) = 0 \text{ and } \sum_{j=1}^{\infty} |B_j| \leq (1+\nu) m_*(E).$$

Problem 2. *Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$.*

Problem 3. *Suppose F is an increasing function on $[a,b]$.*

- *Prove that we can decompose F as $F = F_A + F_C + F_J$, where F_A, F_C, F_J are increasing, F_A is absolutely continuous; F_C is continuous and F_J is a jump function.*
- *Please, also show that F_A, F_C and F_J are uniquely determined for each F up to an additive constant.*

Note, that the above construction is the Lebesgue decomposition of F.