Functions of Real Variables II (62052/72052)
Home WorkS 9 and 10, due on Tuesday, April 28.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Consider $L^p = L^p(\mathbb{R}^d)$ with Lebesgue measure. Let*

$$f_0(x) = \begin{cases} |x|^{-\alpha} & |x| < 1, \\ 0 & |x| \geq 1 \end{cases} \quad \text{and } f_\infty(x) = \begin{cases} 0 & |x| < 1 \\ |x|^{-\alpha} & |x| \geq 1 \end{cases}$$

- $f_0 \in L^p$ *iff* $p\alpha < d$.
- $f_\infty \in L^p$ *iff* $d < p\alpha$.
- *Study the same questions for*

$$f_0(x) = \begin{cases} \frac{|x|^{-\alpha}}{\log(2/|x|)} & |x| < 1, \\ 0 & |x| \geq 1 \end{cases} \quad \text{and } f_\infty(x) = \begin{cases} 0 & |x| < 1, \\ \frac{|x|^{-\alpha}}{\log(2|x|)} & |x| \geq 1. \end{cases}$$

Problem 2. *Let X be a measure space. Using the argument to prove the completeness of $L^p(X)$, show that if the sequence $\{f_n\}$ converges to f in the L^p norm, then a subsequence of $\{f_n\}$ converges to f almost everywhere.*

Problem 3. *Study equality cases in Hölder's and Minkowski inequalities.*

Problem 4. *Suppose (X_1, μ_1) and (X_2, μ_2) are two measure spaces and $p \in [1, \infty]$. Show that if $f(x_1, x_2) : X_1 \times X_2 \to \mathbb{R}$ is a measurable, non-negative function then*

$$\| \int_{X_2} f(x_1, x_2) d\mu_2 \|_{L^p(X_1)} \leq \int_{X_2} \|f(x_1, x_2)\|_{L^p(X_1)} d\mu_2$$

Hint: *You may wait till we talk about linear functionals and duality.*

Problem 5. *Prove that if $f_i \in L^{p_i}(X)$, where X is a measure space $i = 1, \dots, N$, and $\sum \frac{1}{p_1} = 1$, with $p_i \geq 1$, then*

$$\| \prod_{i=1}^{N} f_i \|_{L^1} \leq \prod_{i=1}^{N} \|f_i\|_{L^{p_i}}.$$

Problem 6. *The convolution of f and g on $\mathbb{R}^d$ equipped with Lebesgue measure is defined by*

$$f * g(x) = \int_{\mathbb{R}^d} f(x - y) g(y) dy.$$

- *If $f \in L^p$, $1 \leq p \leq \infty$, and $g \in L^1$, then show that for almost all x the integrant $f(x - y)g(y)$ is integrable in y, hence $f * g$ is well defined. Moreover, $f * g \in L^p$ and*
$$\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}.$$
- *Assume $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p$, $g \in L^q$, then $f * g \in L^\infty$ with*
$$\|f * g\|_{L^\infty} \leq \|f\|_{L^p} \|g\|_{L^q}.$$
- **(Young's Inequality)** *Now consider $p, q, r \in [1, \infty]$ such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. Then*
$$\|f * g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^r}.$$
Hint: *Write $f(y)g(x - y) = f(y)^a g(x - y)^b [f(y)^{1-a} g(x - y)^{1-b}]$, select suitable a and b and use Problem 5, from above.*