Theory of Matrices
Home Work 10, due Thursday, April 13.
Instructor: Prof. Artem Zvavitch

Problem 1. *Let $\varphi\colon V \to W$ be a linear map and let $\alpha \in \mathbb{R}$ be a nonzero scalar. Show that the maps φ and $\alpha\varphi$ have the same kernel and the same image.*
Plan:

- *To prove that the kernels are the same, show that every v that is in the kernel of φ is also in the kernel of $\alpha\varphi$, and vice versa, that every v that is in the kernel of $\alpha\varphi$ is also in the kernel of φ.*
- *To prove that the images are the same, show that every v that is in the image of φ is also in the image of $\alpha\varphi$, and vice versa, that every v that is in the image of $\alpha\varphi$ is also in the image of φ.*

Let us list first some definitions and theorems that we talked about in class (or will talk on Tuesday). Let $B = \{v_1, \dots, v_n\}$ be a basis for a vector space V. Then any vector $v \in V$ can be written in the form $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Then we defined $[v]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

Let $T\colon V \to V$ be a linear operator. We defined the matrix representing T in the basis B to be $[T]_B = [[T(v_1)]_B, \dots, [T(v_n)]_B]$. We checked in class that

$$[T(v)]_B = [T]_B[v]_B.$$

Let $B = \{u_1, \dots, u_n\}$ and $C = \{v_1, \dots, v_n\}$ be two bases for V. We defined $P = [[v_1]_B \dots [v_n]_B]$ and called it *the change of basis matrix from B to C*. Similarly, we defined $Q = [[u_1]_C \dots [u_n]_C]$ and called it *the change of basis matrix from C to B*.

We will prove on TUESDAY that for $v \in V$ we have

$$P[v]_C = [v]_B \text{ and } Q[v]_B = [v]_C.$$

. We already checked that $Q = P^{-1}$. We will prove on Tuesday the change of basis theorem: for a linear operator $T\colon V \to V$ we have

$$[T]_C = P^{-1}[T]_B P.$$

We also observed that if $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by $T(v) = Av$ for some n by n matrix A then if B is the standard basis for $\mathbb{R}^n$ (that is, $B = \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$) then $[T]_B = A$.

Problem 2. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defines by $T(x, y) = (2x + 3y, y - x)$.*

(a) *Find the matrix $[T]_B$ representing T in the standard basis $B = \{v_1, v_2\} = \{(1, 0), (0, 1)\}$.*
(b) *Let $C = \{u_1, u_2\} = \{(1, 3), (2, 1)\}$. Find the matrix $[T]_C$ representing T in the basis C (use the definition of $[T]_C$ only).*
(c) *Find P, the change of basis matrix from B to C.*
(d) *Verify that $[T]_C = P^{-1}[T]_B P$.*
(e) *For $v = (a, b)$, find $[v]_B$. Next, use one of the formulas in the review above to find $[v]_C$.*

Problem 3. *Let $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $C = (1, 1, 0), (1, 2, 1), (1, 4, 2)$, be two bases for $\mathbb{R}^3$.*

(a) *Find P, the change of basis matrix from B to C.*
(b) *Find Q, the change of basis matrix from C to B.*
(c) *Verify that $Q = P^{-1}$.*

Problem 4. *Let $B = \{(1, 1), (2, 1)\}$ and $C = \{(1, 2), (1, 3)\}$ be two bases for $\mathbb{R}^2$.*

(a) *For $v = (a, b) \in \mathbb{R}^2$ find $[v]_B$ and $[v]_C$.*
(b) *Find P, the change of basis matrix from B to C.*
(c) *Find Q, the change of basis matrix from C to B.*
(d) *Verify that for $v = (a, b) \in \mathbb{R}^2$, we have*

$$P[v]_C = [v]_B \text{ and } Q[v]_B = [v]_C.$$

Problem 5. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T((x, y)) = (3x - 2y, x + y)$, and let $B = \{(1, 0), (0, 1)\}$ and $C = \{(2, 1), (3, 2)\}$ be bases for $\mathbb{R}^2$.*

(a) *Find the matrix $[T]_B$ of T relative to B and the matrix $[T]_C$ of T relative to C.*
(b) *Find P, the change of basis matrix from B to C.*
(c) *Write an equation relating $[T]_B$, $[T]_C$, and P, and then verify that the equation is true by calculation.*