Theory of Numbers
Home Work 1, due Thursday, January 22.
Instructor: Prof. Artem Zvavitch

Problem 1. *Prove that*

$$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4},$$

for all $n \in \mathbb{N}$.

Problem 2. *Prove that*

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n},$$

for all $n \in \mathbb{N}$.

Problem 3. *Prove that if* $n \geq 12$ *then* n *can be written as a sum of* $4's$ *and* $5's$. *For example* $23 = 5 + 5 + 5 + 4 + 4 = 3 * 5 + 2 * 4$. *(**Hint**: it may help to do first* $n = 12, 13, 14, 15$ *as special cases and start your inductive proof from* 16.*)*

Problem 4. *Prove, using induction, that for* $n \geq 2$

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}.$$

Problem 5. *Prove that the number* $n^3 + 2n$ *is divisible by* 3 *for any natural number* n.

Problem 6. *Use the Division Algorithm to prove that*

- *The square of any integer is either of the form* $3k$ *or* $3k + 1$.
- *Prove that* $3m^2 - 1$ *is never a perfect square.*

Problem 7. *Use mathematical induction to prove that* $8 | 5^{2n} + 7$.