Theory of Numbers
Home Work 3, due Thursday, February 5.
Instructor: Prof. Artem Zvavitch

Problem 1. *Prove that* $lcm(mb, ma) = mlcm(b, a)$ *for* $a, b, m \in \mathbb{N}$.

Problem 2. *Prove that for any* $a, b, c \in \mathbb{N}$ *we have*

$$lcm(a, b, c) = \frac{abc \cdot gcd(a, b, c)}{gcd(a, b) \cdot gcd(a, c) \cdot gcd(b, c)}$$

Problem 3. *Prove that for any positive integers* $a_1, a_2, \ldots, a_n$*, there exist integers* $x_1, x_2, ..., x_n$*, such that*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = gcd(a_1, a_2, ..., a_n).$$

Problem 4. *Prove that the fraction*

$$\frac{21n + 4}{14n + 3}$$

Is irreducible for any integer n. **(Hint: fraction** $\frac{a}{b}$ **is irreducible iff** $gcd(a, b) = 1$**, it will help you to note that this is iff there are** $x, y \in \mathbb{Z}$ **such that** $ax + by = 1$**.)**.

Problem 5. *If possible, solve (in Integers!) the following equations and describe the set of positive solutions*

- $6x + 51y = 22.$
- $24x + 138y = 18.$
- $158x - 57y = 7.$
- $\frac{2}{3}x + \frac{1}{2}y = 5.$

Problem 6. *Find the number of men, women, and children in a company of* 20 *persons if together they pay* 20 *coins, each man paying* 3*, each woman* 2*, and each child* $\frac{1}{2}$.

A POSSIBLE HINTS FOR PROBLEM 2: You may prove it directly using properties of max and min or

Step 1 Prove $gcd(a, b, c) = gcd(gcd(a, b), gcd(a, c))$. This can be done using the definition of gcd of playing with prime decomposition and max and min functions.

Step 2 Prove $lcm(a, lcm(b, c)) = lcm(a, b, c)$. This can be done using the definition of gcd of playing with prime decomposition and max and min functions.

Step 3 Prove $gcd(a, lcm(b, c)) = lcm(gcd(a, b), gcd(a, c))$. See lecture notes or just prove it yourself using prime decomposition and max and min functions.

Now use $lcm(x, y)gcd(x, y) = xy$ for $x = gcd(a, b)$ and $y = gcd(a, c)$):

$$\frac{abc \cdot gcd(a, b, c)}{gcd(a, b) \cdot gcd(a, c) \cdot gcd(b, c)} == \frac{abc \cdot gcd(a, b, c)}{gcd(gcd(a, b), gcd(a, c)) \cdot lcm(gcd(a, b), gcd(a, c)) \cdot gcd(b, c)}$$

use step 2:

$$= \frac{abc}{lcm(gcd(a, b), gcd(a, c)) \cdot gcd(b, c)} = \frac{a \cdot lcm(b, c)}{lcm(gcd(a, b), gcd(a, c))}$$

$$= \frac{lcm(a, lcm(b, c)) \cdot gcd(a, lcm(b, c))}{lcm(gcd(a, b), gcd(a, c))}$$

Use step 2 and step 3 to finish.