Theory of Numbers (47011/57011)
Final Homework
Instructor: Prof. Artem Zvavitch
Due in the day of final exam. Each problem is 7 points.

Problem 1. *Prove that of* $a \in \mathbb{N}$ *then* $360 \big| a^2(a^2-1)(a^2-4)$.

Problem 2. *Consider* $a, b, c \in \mathbb{N}$. *Prove that if* $gcd(a,b) = gcd(a,c) = 1$ *then* $gcd(a,bc) = 1$.

Problem 3. *Consider* $a, b \in \mathbb{N}$ *such that* $gcd(a,b) = lcm(a,b)$ *prove that then* $a = b$.

Problem 4. *Divide* 100 *into two summands such that one is divisible by* 7 *and another by* 11.

Problem 5. *Show that any composite three-digit number must have a prime factor which is less then* 31.

Problem 6. *Assume that* p_n *is the* n*th prime number. Prove that* $p_n > 2n-1$ *for* $n \geq 5$.

Problem 7. *Prove that if* $gcd(a,n) = 1$ *then the integers*

$$c, c+a, c+2a, \dots, c+(n-1)a$$

form a complete set of residues modulo n *for any* c. *Would this statement be still true without assumption of* $gcd(a,n) = 1$*?*

Problem 8. *Determine the last three digits of number* 7^{999}.

Problem 9. *Sole the linear congruence* $17x \equiv 3 \ (mod\ 2 \cdot 3 \cdot 5 \cdot 7)$.

Problem 10. *Find the solution of the following system of equations*

$$11x + 5y \equiv 7(mod\ 20) \ \text{and} \ 6x + 3y \equiv 8(mod\ 20).$$

Problem 11. *Assume* p *is odd prime, use Fermat's theorem to show that*

$$1^p + 2^p + 3^p + \cdots + (p-1)^p \equiv 0(mod\ p).$$

Problem 12. *Use Wilson's theorem to show that for any odd prime* p*:*

$$1^2 \cdot 3^2 \cdot 5^2 \dots (p-2)^2 \equiv (-1)^{(p+1)/2}(mod\ p)$$

It may help to note that $k = -(p-k) \ (mod\ p)$.

Problem 13. *Let* f *and* g *be multiplicative functions such that* $f(p^k) = g(p^k)$ *for each prime* p *and* $k \geq 1$. *Prove that* $f(n) = g(n)$ *for all* $n \in \mathbb{N}$.

Problem 14. *Prove that* $\phi(n^2) = n\phi(n)$, *for all* $n \in \mathbb{N}$.

Problem 15. *Prove that odd prime devisors of the integer* $n^4 + 1$ *are of the form* $8k+1$.

Problem 16. *Find primitive roots of* 41 *and* 82.