Asymptotic Theory of Finite Dimensional Normed Spaces.
Home Works 1 and 2, due Monday September 20
Instructor: Prof. Artem Zvavitch

Problem 1. *Let $K \in \mathbb{R}^n$ be a symmetric convex body, and $u^\perp = \{x \in \mathbb{R}^n : x \cdot u = 0\}$, for $u \in S^{n-1}$. Prove that*

$$(K|u^\perp)^\circ = K^\circ \cap u^\perp,$$

where $K|u^\perp$ is the orthogonal projection of K on $u^\perp$. Please, note that $(K|u^\perp)^\circ$ is **the dual body to $K|u^\perp$ in $u^\perp$**.

Problem 2. *Let $K, L, M \subset \mathbb{R}^n$, be convex, symmetric bodies.*

- *Prove that $d_{BM}(K,L) \le d_{BM}(K,M)d_{BM}(M,L)$.*
- *Compare $d_{BM}(K,L)$ and $d_{BM}(K^\circ, L^\circ)$.*

Problem 3. *Prove that*

$$(B_p^n)^\circ = B_q^n, \text{ for all },p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,$$

where $B_p^n = \{x \in \mathbb{R}^n : \sum\limits_{i=1}^{n} |x_i|^p \le 1\}$.

Problem 4. *Find $\mathrm{vol}(B_\infty^n)$, where vol is a regular Volume measure (Lebesque measure on $\mathbb{R}^n$). Now try to find $\mathrm{vol}(B_1^n)$. Finally try to find $\mathrm{vol}(B_p^n)$ for $0<p<\infty$.(HINT: consider the following integral*

$$I_p = \int_{\mathbb{R}^n} e^{-\|x\|_p^p} dx.$$

First notice that it is easy to compute I_p, just separating the variables. Yes, you will get Gamma functions. Next show that

$$I_p = \int_0^\infty p t^{p-1} e^{-t^p} \mathrm{vol}\{x : \|x\|_p < t\} dt.$$

Now "glue" those two computations to find the formula.)

Problem 5. *Santalo inequality claims that $\mathrm{vol}(K)\mathrm{vol}(K^\circ) \le \mathrm{vol}(B_2^n)^2$ for all convex symmetric bodies $K \subset \mathbb{R}^n$ (i.e. maximum of the volume product of K and K° is achieved for $K = B_2^n$). It is interesting that the minimal case is still an open problem!!! it is so called Mahler conjecture:*

$$\mathrm{vol}(B_1^n)\mathrm{vol}(B_\infty^n) < \mathrm{vol}(K)\mathrm{vol}(K^\circ).$$

Please, confirm Santalo inequality and Mahler conjecture in the special case of $K = B_p^n$, $p>1$.

Problem 6. *Consider and ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, find $\mathrm{vol}(\mathcal{E})\mathrm{vol}(\mathcal{E}^\circ)$.*