

Introduction to Partial Differential Equations.
Home Work 1, due Wednesday, February 4.
Instructor: Prof. Artem Zvavitch

Problem 1. Consider polar coordinates (r, θ) , where r is the distance from the origin, θ is the angle in xy -plane from x -axis.

- Show that the standard coordinates can be rewritten as

$$x(r, \theta) = r \cos \theta,$$

$$y(r, \theta) = r \sin \theta,$$

- Notice that $r^2 = x^2 + y^2$ apply to this equality $\frac{\partial}{\partial x}$ to find $\frac{\partial r}{\partial x}$. Also find $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$.
- Consider function $u(x, y)$. Use the chain rule to prove that

$$\frac{\partial u}{\partial x} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta}.$$

- Now you are ready to compute the Laplacian in polar coordinates, prove that

$$\nabla^2 u = \Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

Problem 2. Show that the following operators are linear:

- $Lu = \Delta u$.
- $Lu = \Delta [K_0(x, y, z)u(x, y, z)]$.
- $Lu = \int_0^x u(y) dy$.

Problem 3. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

- $\Delta u(x, y) = 0$.
- $\frac{\partial u}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right)$

Problem 4. Solve heat equation $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ subject to $u(0, t) = 0$, $u(L, t) = 0$ and $u(x, 0) = 3 \sin \frac{\pi x}{L} - 2 \sin \frac{4\pi x}{L}$.