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A REMARK ON THE EXTREMAL NON-CENTRAL SECTIONS OF THE UNIT CUBE.

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Abstract. In this paper, we investigate extremal volumes of non-central slices of the unit cube. The case of central hyperplane sections is known and was studied in [Ha], [Ba1] [He1]. The case of non-central sections, i.e. when we dictate that the hyperplane must be a certain distance $t > 0$ from the center of the cube, is open in general and the same is true about sections of the unit cube by slabs, with some partial results provided in [BK], [KK1], [KK2]. In this paper we give a full solution for extremal one-dimensional sections and a partial solution for extremal hyperplane slices for the case $t > \frac{\sqrt{n-1}}{2}$. We also make a remark on minimal volume slices of the cube by slabs of width $2t$, when $t > \frac{\sqrt{n-1}}{2}$.

1. Introduction

We denote by $Q_n = [-1/2, 1/2]^n$ an n-dimensional cube of volume 1. As usual, we denote by $x \cdot y$ the inner product of two vectors $x, y \in \mathbb{R}^n$ and by $|x|$ the length of vector $x \in \mathbb{R}^n$. Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be a Euclidean unit sphere, $t\mathbb{S}^{n-1}$ be a sphere of radius t, $d(A, x) = \inf_{a\in A} |x - a|$ be the distance from a point $x \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$ and $\operatorname{diam}(A) = \sup\{|x - y| : x, y \in A\}$ be the diameter of $A \subset \mathbb{R}^n$. We write $|A|$ for the k-dimensional Lebesgue measure (volume) of a measurable set $A \subset \mathbb{R}^n$, where $k = 1, \dots, n$ is the dimension of the minimal flat containing A. For $a, b \in \mathbb{R}^n$, we denote by $[a, b]$ the segment joining a to b: $[a, b] = \{(1-t)a + tb : t \in [0, 1]\}$. A *convex body* is a compact convex subset of $\mathbb{R}^n$ with nonempty interior and Q is *symmetric* if it is centrally symmetric with center at the origin, i.e. $Q = -Q$. We refer to [K] or [Ga1] for an excellent introduction to the subject.

In this paper we study the extremal volumes of slices of Q_n. The original question posed concerns slices by $(n-1)$-dimension subspaces H of $\mathbb{R}^n$: What are the minimal and maximal values of $|Q_n \cap H|$? It was shown by Hadwiger [Ha] that the section of minimal volume is the canonical section (i.e. the section orthogonal to one of the canonical unit vectors). The solution to the maximal case was conjectured by Hensley [He1]. Ball [Ba1] (see also [NP] and [K]) proved Hensley's conjecture by showing that the maximal volume of hyperplane sections of Q_n is $\sqrt{2}$ and it attains for the hyperplane H with normal vector $\xi = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \dots, 0)$.

Hensley was motivated by a more general statement, which was conjectured by Good. The claim stated that, for any k-dimensional subspace P_k of $\mathbb{R}^n$ with $k < n$, $|P_k \cap Q_n| \geq 1$. Ball's proof addressed the case when $k = n - 1$; Vaaler gave a full proof of Good's conjecture [Va] (see also [Zo]).

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The above results were further extended by Oleszkiewicz and Pełczyński [OP], who proved an analog to Ball's theorem in the complex space $\mathbb{C}^n$. Ball's result was also generalized to the case of Gaussian and Gaussian-type measures ([BGMN], [Zv] and [KK3]).

Much less is known about non-central hyperplane sections (i.e. sections of Q_n by $\{x \in \mathbb{R}^n : x \cdot \xi = t\}$, for a fixed $t > 0$) and sections of the cube by slabs (i.e. sections of Q_n by $\{x \in \mathbb{R}^n : |x \cdot \xi| \leqslant t\}$, for a fixed $t > 0$). Barthe and Koldobsky [BK] provided a number of interesting results on sections of Q_n by slabs, including the complete solutions of the two dimensional case and the case of slabs of small width. The work was continued in [KK1] and [KK2], but the general question of finding extremal, non-central sections/slabs is still open.

In Section 2, below, we give the optimal lengths of intersection between Q_n and a line ℓ that is a specified distance $t > 0$ away from the origin. Next, in Section 3, we provide a partial solution to the initial hyperplane problem, for when $t > \frac{\sqrt{n-1}}{2}$ and also describe the extremal intersection with slabs in this case. Although the methods of solution are rather cumbersome, we hope that the data could be useful to help predict the results of other related problems.

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2. One dimensional Slices

2.1. **The case of $\mathbb{R}^2$.**

Theorem 1. *For a line ℓ that is distance $t \in [0, 1/\sqrt{2}]$ away from the origin, we have that*

$$\min_{\ell \subset \mathbb{R}^2, d(\ell,0)=t} |Q_2 \cap \ell| = \begin{cases} 1, & t \in [0, \frac{1}{2}(\sqrt{2}-1)], \\ \sqrt{2} - 2t, & t \in [\frac{1}{2}(\sqrt{2}-1), \frac{1}{2}], \\ 0, & t \in (\frac{1}{2}, \frac{1}{\sqrt{2}}], \end{cases}$$

and

$$\max_{\ell \subset \mathbb{R}^2, d(\ell,0)=t} |Q_2 \cap \ell| = \begin{cases} \frac{2}{\sqrt{2-(2t)^2}+2t}, & t \in [0, \frac{1}{2}], \\ \frac{1}{2}\left(2t - \sqrt{4t^2-1}\right), & t \in (\frac{1}{2}, \frac{3}{4}\frac{1}{\sqrt{2}}], \\ \sqrt{2} - 2t, & t \in (\frac{3}{4}\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \end{cases}.$$

Proof. The $\mathbb{R}^2$ case is done by direct calculations, which are very similar to the computation of extremal slab provided in [BK]. For the sake of completeness we will sketch it here.

Let ℓ be a line tangent to $t\mathbb{S}^1$. Take the standard polar parametrization and let $(t\cos\theta, t\sin\theta)$ be the tangent point of ℓ and $t\mathbb{S}^1$, where θ is the angle from the positive x-axis. Thus, ℓ is defined by an equation $x\cos\theta + y\sin\theta = t$.

Due to the symmetries of the cube, we need only consider $\theta \in [0, \frac{\pi}{4}]$. We note that for $t > \frac{1}{2}$, we have the trivial result for the minimal case $|Q_2 \cap \ell| = 0$ when we take $\theta = 0$. We will start with $t \leq \frac{1}{2}$ and will find the maximal interval for $t > \frac{1}{2}$ in the end.

For $\theta \in [0, \frac{\pi}{4}]$, ℓ must intersect ∂Q_2 by the edge $\{|x| \leq \frac{1}{2}, y = \frac{1}{2}\}$. We notice that ℓ will intersect the opposite edge of Q_2 (i.e. $\{|x| \leq \frac{1}{2}, y = -\frac{1}{2}\}$) when $\theta \in [0, \frac{\pi}{4} - \arcsin(\sqrt{2}t)]$, yielding $|\ell \cap Q_2| = \sec\theta$, which is an increasing function on the prescribed interval. On the other hand, ℓ will intersect $\{x = \frac{1}{2}, |y| \leq \frac{1}{2}\}$ edge of Q_2, when $\theta \in [\frac{\pi}{4} - \arcsin(\sqrt{2}t), \frac{\pi}{4}]$, yielding

$$\begin{aligned} f_t(\theta) = |Q_2 \cap \ell| &= |(\frac{t - \frac{1}{2}\sin\theta}{\cos\theta}, \frac{1}{2}) - (\frac{1}{2}, \frac{t - \frac{1}{2}\cos\theta}{\sin\theta})| \\ &= \frac{|t - \frac{1}{2}\sin\theta - \frac{1}{2}\cos\theta|}{\sin\theta\cos\theta} \\ &= \sqrt{2}\frac{\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t}{\sin 2\theta}. \end{aligned}$$

Next we need to compute $f_t'(\theta)$:

$$\begin{aligned} f_t'(\theta) &= \sqrt{2}\frac{\cos\left(\theta + \frac{\pi}{4}\right)\sin 2\theta - 2\cos 2\theta(\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t)}{\sin^2 2\theta} \\ &= \sqrt{2}\frac{\cos\left(\theta + \frac{\pi}{4}\right)\sin 2\theta - 2\sin(\frac{\pi}{2} + 2\theta)(\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t)}{\sin^2 2\theta} \\ &= \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin^2 2\theta}\left[\sin 2\theta - 4\sin(\frac{\pi}{4} + \theta)(\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t)\right] \\ &= \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin^2 2\theta}\left[-\cos(\frac{\pi}{2} + 2\theta) - 4\sin^2(\frac{\pi}{4} + \theta) + 4\sqrt{2}t\sin\left(\theta + \frac{\pi}{4}\right)\right] \\ &= \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin^2 2\theta}\left[-2\sin^2(\frac{\pi}{4} + \theta) + 4\sqrt{2}t\sin\left(\theta + \frac{\pi}{4}\right) - 1\right]. \end{aligned}$$

The last expression is nonpositive for $\theta \in [0, \pi/4]$ and $t \in [0, 1/2]$ (indeed, $32t^2 - 8 \leqslant 0$). Thus, $f_t(\theta)$ is a decreasing function on the required interval having the following critical values:

- For $t \in [0, \frac{1}{2}(\sqrt{2} - 1)]$, the minimum value of $|Q_2 \cap \ell| = 1$ occurs when $\theta = 0$.
- For $t \in [\frac{1}{2}(\sqrt{2} - 1), \frac{1}{2}]$, the minimum of $|Q_2 \cap \ell| = \sqrt{2} - 2t$ occurs when $\theta = \frac{\pi}{4}$.
- For $t \in (\frac{1}{2}, \frac{1}{\sqrt{2}}]$, the minimum of $|Q_2 \cap \ell| = 0$ occurs when $\theta = 0$.
- For $t \in [0, \frac{1}{2}]$, the maximum of $|Q_2 \cap \ell| = \frac{2}{\sqrt{2-(2t)^2+2t}}$ occurs when $\theta = \frac{\pi}{4} - \arcsin t\sqrt{2}$, i.e. when ℓ goes through a vertex of the square to an edge containing the opposite vertex.

Finally, for $t \in (\frac{1}{2}, \frac{1}{\sqrt{2}}]$, we notice that ℓ will intersect Q_2 for $\theta \in [\arcsin(\sqrt{2}t) - \frac{\pi}{4}, \frac{\pi}{4}]$ (i.e. $\sin\left(\theta + \frac{\pi}{4}\right) \in [\sqrt{2}t, 1]$) and must intersect two adjacent edges, yielding

$$f_t(\theta) = |Q_2 \cap \ell| = \sqrt{2}\frac{\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t}{\sin 2\theta}.$$

From above we get

$$f_t'(\theta) = \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin^2 2\theta}\left[-2\sin^2(\frac{\pi}{4} + \theta) + 4\sqrt{2}t\sin\left(\theta + \frac{\pi}{4}\right) - 1\right],$$

which is positive if $\sin\left(\theta + \frac{\pi}{4}\right) \in [\sqrt{2}t - \frac{1}{2}\sqrt{8t^2 - 2}, \sqrt{2}t + \frac{1}{2}\sqrt{8t^2 - 2}]$ and negative otherwise. Thus, $f_t(\theta)$ achieves the maximal value at

$$\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}t + \frac{1}{2}\sqrt{8t^2 - 2} \quad \text{if} \quad \sqrt{2}t + \frac{1}{2}\sqrt{8t^2 - 2} \leqslant 1,$$

and at $\theta = \pi/4$ otherwise.

We notice that $\sqrt{2}t + \frac{1}{2}\sqrt{8t^2-2} \leqslant 1$ gives $t \in (\frac{1}{2}, \frac{3}{4}\frac{1}{\sqrt{2}}]$. From $\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}t + \frac{1}{2}\sqrt{8t^2-2}$ we get

$$\begin{aligned}\sin 2\theta = -\cos(\frac{\pi}{2} + 2\theta) = 2\sin^2(\frac{\pi}{4} + \theta) - 1 &= 8t^2 + 2\sqrt{2}t\sqrt{8t^2-2} - 2 \\ &= \sqrt{8t^2-2}\left(\sqrt{8t^2-2} + 2\sqrt{2}t\right).\end{aligned}$$

Thus, the length of maximal section, for this case, is

$$|Q_2 \cap \ell| = \sqrt{2}\frac{\frac{1}{2}\sqrt{8t^2-2}}{\sqrt{8t^2-2}\left(\sqrt{8t^2-2} + 2\sqrt{2}t\right)} = \frac{1}{2\left(\sqrt{4t^2-1} + 2t\right)} = \frac{1}{2}\left(2t - \sqrt{4t^2-1}\right).$$

Finally maximal length of intersection for $t \in [\frac{3}{4}\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ occurs when $\theta = \dfrac{\pi}{4}$, and the length, as mentioned before, is $\sqrt{2} - 2t$.

□

2.2. **Minimal Case.**

Theorem 2. *For a line ℓ that is distance t away from the origin, we have that*

$$\min_{\ell \subset \mathbb{R}^n, d(\ell,0)=t} |Q_n \cap \ell| = \min_{\ell \subset \mathbb{R}^2, d(\ell,0)=t} |Q_2 \cap \ell| = \begin{cases} 1, & t \in [0, \frac{1}{2}(\sqrt{2}-1)] \\ \sqrt{2} - 2t, & t \in [\frac{1}{2}(\sqrt{2}-1), \frac{1}{2}] \\ 0, & t \in (\frac{1}{2}, \infty) \end{cases}.$$

Proof. Our goal is to show that $\mathbb{R}^n$ case is equivalent to the case of the square. We again remark the case $t > 1/2$ is trivial. We also note that by Brunn's theorem (see [Ga1] or [Ga2])

$$\min_{\ell \subset \mathbb{R}^n, d(\ell,0)=t} |\ell \cap Q_n|$$

is a decreasing function for $t \geqslant 0$.

Let $[a, b] = Q_n \cap \ell$ be segment of minimal length among lines ℓ which are at distance $t \leqslant 1/2$ from the origin. Notice that from $d([a, b], 0) \leqslant 1/2$, we get that $[a, b]$ can not belong to an $(n-2)$-dimensional face of the cube.

Note that $[a, b]$ must pass through two distinct faces of Q_n. By symmetry, we can assume that one face is given by $\{(\frac{1}{2}, x_2, x_3, \dots, x_n) : |x_i| \leq \frac{1}{2}\}$. Then the other face is either opposite to the first, i.e. $\{(-\frac{1}{2}, x_2, x_3, \dots, x_n) : |x_i| \leq \frac{1}{2}\}$, or by symmetry again, we can reorient so that it is $\{(x_1, \frac{1}{2}, x_3, x_4, \dots, x_n) : |x_i| \leq \frac{1}{2}\}$.

Consider the orthogonal projection of Q_n and $[a, b]$ onto the x_1x_2-coordinate plane. The cube will simply project down to Q_2, and $[a, b]$ will project to $[a', b']$, with $|b' - a'| \leqslant |b - a|$. Notice that $[a', b']$ passes either through opposite sides of Q_2 (the first case above), or adjacent sides of Q_2 (the second case). We notice that $[a', b']$ must be a distance $t' \leq t$ away from the origin. Since we've already established that $\min |Q_2 \cap \ell|$ is a non-increasing function of t, we get that there is a line ℓ contained in x_1x_2-coordinate plane and tangent to $t\mathbb{S}^{n-1}$ such that $|b' - a'| \geq |Q_2 \cap \ell|$.

Finally, for $[a, b]$ to be minimal, we must have equalities in above inequalities, i.e.

$$|b - a| = |b' - a'| \text{ and } t = t',$$

which immediately gives that $[a,b]$ must belong to a two dimensional subspace parallel to a two dimensional coordinate plane. $\square$

2.3. **Maximal Case.** For the maximal case, we will first describe our results in geometric terms, providing numerical answers at the end of this section in Theorem 4.

Theorem 3. *For a line $\ell \subset \mathbb{R}^n$ that is distance $t \in [0, \frac{1}{2}]$ away from the origin, the maximal length $|Q_n \cap \ell|$ occurs when ℓ travels through a vertex of Q_n and passes through an edge containing the opposite vertex. For $t \in [\frac{1}{2}, \frac{\sqrt{n}}{2}]$, the maximal ℓ is confined to a face of Q_n.*

The following elementary lemmas are needed for the proof of the Theorem 3:

Lemma 1. *Consider a convex bounded polytope $P \subset \mathbb{R}^m$, let ℓ be a line in $\mathbb{R}^m$. Then the maximal length $|P \cap \ell|$ occurs when ℓ is passing through two vertices of P.*

Proof. The lemma is an immediate consequence of the property of convex functions over convex domain. Indeed, assume that $[a,b] = P \cap \ell$ is maximal and that a is not a vertex of P. Then, $f(x) = |x-b|$ is a convex function and thus must achieve its maximal value at an extremal point of P. $\square$

Remark 1. *Note that the above lemma is not true if we require the lines to pass through a fixed point in P and would not ask for P to be symmetric.*

Lemma 2. *Let $Q \subset \mathbb{R}^n$ be a convex symmetric polytope and let $\ell \subset \mathbb{R}^n$ be a line that is distance $t < \frac{1}{2}\mathrm{diam}(Q)$ from the origin. The maximal length of $[a,b] = Q \cap \ell$ occurs when a, b belong to the edges of Q.*

Proof. Assume that $[a,b] = Q \cap \ell$, is maximal, but at least one of the endpoints a, b does not belong to an edge of Q. Let $\ell \cap t\mathbb{S}^{n-1} = t\xi$, where $\xi \in \mathbb{S}^{n-1}$. Consider an affine hyperplane $H(\xi, t) = \{x \in \mathbb{R}^n : x \cdot \xi = t\}$ containing ℓ and tangent to $t\mathbb{S}^{n-1}$ at $t\xi$. Let $P = Q \cap H$, then $[a,b]$ is not a maximal segment inside P, otherwise a, b would be vertices of P and thus would each belong to edges of Q, by Lemma 1. Assume $[x,y]$ is a maximal segment inside P. If $t\xi \in [x,y]$ then this contradicts the maximality of $[a,b]$. Otherwise $[x,y]$ is outside of $t\mathbb{S}^{n-1}$. Next we consider a two-dimensional subspace M containing $[x,y]$. We note that $Q \cap M$ is a convex symmetric body and $[x,y]$ is a one-dimensional section of $Q \cap M$ at distance greater than t from the origin. Brunn's theorem tells us that there is a one-dimensional slice of $Q \cap M$ (parallel to $[x,y]$) of length no less than the length of $[x,y]$ and tangent to $t\mathbb{S}^1$, which again contradicts the maximality of $[a,b]$. $\square$

Proof of Theorem 3: It follows from Lemma 2 that the segment at distance t of maximal length must pass through the edges of Q_n.
Case 0: The two edges share a vertex. Then the maximal segment must belong to a face of Q_n and $t \geqslant \frac{1}{2}$.
Case I: The two edges are parallel.

We first note that if $t < 1/2$ then the parallel edges must be symmetric to each other with respect to the origin (if this is not the case then the edges will be contained in a face of Q_n and then any segment connecting them will be at the distance greater or equal then $1/2$ from the origin).

We consider the intersection of Q_n and the two-dimensional subspace defined by these edges. The result is that our line ℓ passes through two opposite of edges of side length 1 in a

1 by $\sqrt{n-1}$ rectangle, and is tangent to a circle of radius t that has its center at the center of the rectangle. We can set this rectangle on the coordinate axes so that the center of the rectangle and the circle is at the origin, and the rectangle has vertices $(\pm\frac{1}{2}, \pm\frac{\sqrt{n-1}}{2})$, ℓ thus intersects the boundary of this rectangle at the points $(x_0, \frac{\sqrt{n-1}}{2})$ and $(y_0, -\frac{\sqrt{n-1}}{2})$.

As with our two-dimensional minimum problem, we can use basic calculus to solve for the optimal length. Again, let θ be the angle between the positive x-axis and the line perpendicular to ℓ. We need only consider $\theta \in \left[0, \arcsin\frac{1}{\sqrt{n}} - \arcsin\frac{2t}{\sqrt{n}}\right]$. Our length of intersection is a function of θ, $|Q_n \cap \ell| = \sqrt{n-1}\sec\theta$. We find that the maximum of $|Q_n \cap \ell|$ occurs at $\theta = \arcsin\frac{1}{\sqrt{n}} - \arcsin\frac{2t}{\sqrt{n}}$ and the maximum value is

$$|Q_n \cap \ell| = \frac{n\sqrt{n-1}}{\sqrt{(n-1)(n-4t^2)} + 2t}. \tag{1}$$

We also note that the maximal ℓ will intersect the boundary of the rectangle at the vertex (which is also a vertex of Q_n), which finishes the proof of this case.

Finally, we note that if $t \geqslant 1/2$, then the segment connecting parallel edges must belong to a face of Q_n.

Case II: The two edges are skew.

We first notice that if edges are contained in the same face then, clearly, $t \geqslant 1/2$. Thus, we only need to consider the case when two edges are not contained in the same face, which gives that the distance between them is $\sqrt{n-2}$. Note that t could be greater than 1/2 in this case, thus we will need to make sure that the maximal edge will be forced to the face of Q_n for $t \geqslant 1/2$ and endpoints belong to the skew edges.

We remark that in this case the edges must belong to a 3 dimensional subspace of $\mathbb{R}^n$. Consider the three-dimensional section of Q_n defined by those edges. The result is a 1 by 1 by $\sqrt{n-2}$ rectangular prism. The section will contain the origin and its intersection with $t\mathbb{S}^{n-1}$ is $t\mathbb{S}^2$. Embedding the prism in to $\mathbb{R}^3$ with its center at the origin, we get a rectangular prism with vertices $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{n-2}}{2})$. The line ℓ intersects the boundary of the prism at $(x_0, \frac{1}{2}, \frac{\sqrt{n-2}}{2})$, and at $(-\frac{1}{2}, y_0, -\frac{\sqrt{n-2}}{2})$ (or at $(\frac{1}{2}, y_0, -\frac{\sqrt{n-2}}{2})$, which, by the symmetry, is an equivalent case).

We immediately notice that if $t > 1/\sqrt{2}$, this case becomes impossible; there is no segment connecting the aforementioned edges of our prism which is also tangent to $t\mathbb{S}^2$. In fact, all such segments will intersect $t\mathbb{S}^2$. Thus, we concentrate on the case when $t \in [0, \frac{1}{\sqrt{2}}]$.

Let $D(x,y)$ be the distance between $(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2})$ and $(\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2})$. We use the method of Lagrange multipliers to optimize the function

$$D^2 = (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + n - 2$$

with the constraints that $x, y \in [-\frac{1}{2}, \frac{1}{2}]$ and the line ℓ is at distance t away from the origin. That is,

$$\frac{|(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2}) \times (\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2})|}{D} = t,$$

or

$$|(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2}) \times (\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2})|^2 - D^2t^2 = 0. \tag{2}$$

To find possible extremal points, we consider the function

$$\begin{aligned} F_\lambda =& D^2 - \lambda \left(|(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2}) \times (\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2})|^2 - D^2 t^2 \right) \\ =& (1+\lambda t^2) \left((x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 + n - 2 \right) \\ & \qquad - \lambda \left(\frac{n-2}{4}(y+\frac{1}{2})^2 + \frac{n-2}{4}(x+\frac{1}{2})^2 + (xy-\frac{1}{4})^2 \right). \end{aligned}$$

Computing partial derivatives, we get

$$\begin{aligned} \frac{\partial F_\lambda}{\partial x} &= 2(1+\lambda t^2)(x-\frac{1}{2}) - \lambda \left(\frac{n-2}{2}(x+\frac{1}{2}) + 2y(xy-\frac{1}{4}) \right), \\ \frac{\partial F_\lambda}{\partial y} &= 2(1+\lambda t^2)(y-\frac{1}{2}) - \lambda \left(\frac{n-2}{2}(y+\frac{1}{2}) + 2x(xy-\frac{1}{4}) \right). \end{aligned}$$

We now set $\frac{\partial F_\lambda}{\partial x} = \frac{\partial F_\lambda}{\partial x} = 0$ and solve $(y-\frac{1}{2})\frac{\partial F_\lambda}{\partial x} - (x-\frac{1}{2})\frac{\partial F_\lambda}{\partial x} = 0$, which, after simplification, becomes

$$(y-x)\left(\frac{n-2}{2} + (xy-\frac{1}{4})(2(y+x)-1) \right) = 0.$$

We notice that $(xy-\frac{1}{4})(2(y+x)-1) > -\frac{1}{2}$, for $x, y \in [-\frac{1}{2}, \frac{1}{2}]$. Thus, the only possible solution of the above equation is $x = y$. In this case (2) becomes

$$\frac{n-2}{2}(x+\frac{1}{2})^2 + (x^2-\frac{1}{4})^2 = t^2\left(2(x-\frac{1}{2})^2 + n - 2 \right)$$

or

$$\frac{(x+\frac{1}{2})^2}{2}\left(n-2+2(x-\frac{1}{2})^2 \right) = t^2\left(2(x-\frac{1}{2})^2 + n - 2 \right)$$

and using $|x| \leqslant \frac{1}{2}$ and $t \in [0, \frac{1}{\sqrt{2}}]$ we get one possible critical value $x = -\frac{1}{2} + \sqrt{2}t$ and the corresponding segment length

$$D = \sqrt{2(1-\sqrt{2}t)^2 + n - 2} = \sqrt{4t^2 - 4\sqrt{2}t + n}. \tag{3}$$

Now we will show that (3) does not represent the global maximum. For $t \in [0, \frac{1}{2}]$ we should compare (3) with the parallel edges case (1). We will show that (1) $\geqslant$ (3). To do so we square both quantities and take the difference. Thus, we need to show that

$$n^2(n-1) - (4t^2 - 4\sqrt{2}t + n)(4t^2 + 4t\sqrt{(n-1)(n-4t^2)}) + (n-1)n - 4t^2(n-1)) > 0.$$

We simplify the above inequality and divide by $-4t$ and now need to show that

$$(4t^2 - 4\sqrt{2}t + n)\sqrt{(n-1)(n-4t^2)} + nt - 4\sqrt{2}n(n-1) - 4(n-2)t^3 + 4\sqrt{2}(n-2)t^2 < 0.$$

Using that $n \geqslant 3$ and $t \in [0, \frac{1}{2}]$ and writing the maximal value for each of the above summands, it is enough to show that

$$n\sqrt{(n-1)n} + \frac{n}{2} - 4\sqrt{2}n(n-1) + \sqrt{2}(n-2) < 0,$$

which is true for $n \geqslant 3$.

Finally (3) must be compared with the boundary values at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ (or the symmetric cases $y = -\frac{1}{2}$ and $y = \frac{1}{2}$). We notice that we get exactly the same answer as in parallel edges case when $x = -\frac{1}{2}$ (because we get a vertex and an edge containing the opposite vertex), again with the restriction that $t \in [0, \frac{1}{2}]$.

Next we treat the case $x = \frac{1}{2}$. Note that in this case $Q_n \cap \ell$ will belong to a face of our prism (and thus, to a face of Q_n) so we know that $t \geqslant 1/2$. We claim that this is the maximal case for $t \geqslant 1/2$. To prove this claim we only need to show that the length of the segment we get here is greater than (3), with restriction $t \in [1/2, 1/\sqrt{2}]$.

To calculate the length of $Q_n \cap \ell$, we consider the the rectangular 1 by $\sqrt{n-2}$ face of our prism that contains $Q_n \cap \ell$. We note that this face intersects $t\mathbb{S}^2$ and creates $r\mathbb{S}^2$, where $r = \sqrt{t^2 - \frac{1}{4}} \leqslant 1/2$, using a calculation that is similar to the $\mathbb{R}^2$ case and computations from Case I (i.e., we substitute $n-1$ instead of n in (1)). This yields

$$\frac{(n-1)\sqrt{n-2}}{\sqrt{(n-2)(n-1-4r^2)}+2r} = \frac{(n-1)\sqrt{n-2}}{\sqrt{(n-2)(n-4t^2)}+2\sqrt{t^2-\frac{1}{4}}}. \tag{4}$$

We claim that above is greater than (3). To show this we must verify that

$$(n-1)^2(n-2) - (2(1-\sqrt{2}t)^2 + n - 2)(\sqrt{(n-2)(n-4t^2)} + \sqrt{4t^2-1})^2 \geqslant 0,$$

which, after simplification, becomes

$$\begin{aligned}(1-\sqrt{2}t)^2\Big((n-1)^2 - 2 - 4(n-3)t^2 + 2\sqrt{4t^2-1}\sqrt{(n-2)(n-4t^2)}\Big)\\ -(n-2)\left(1 + 2(n-3)t^2 - \sqrt{4t^2-1}\sqrt{(n-2)(n-4t^2)}\right) \leqslant 0,\end{aligned}$$

and after division by $(1-\sqrt{2}t)^2$ becomes

$$\begin{aligned}\Big((n-1)^2 - 2 - 4(n-3)t^2 + 2\sqrt{4t^2-1}\sqrt{(n-2)(n-4t^2)}\Big)\\ -\frac{(n-2)(n-1)^2(\sqrt{2}t+1)^2}{\Big(1 + 2(n-3)t^2 + \sqrt{4t^2-1}\sqrt{(n-2)(n-4t^2)}\Big)} \leqslant 0.\end{aligned}$$

Further simplification shows that it is enough to prove that

$$\Big(1 + 2(n-3)t^2 + \sqrt{4t^2-1}\sqrt{(n-2)(n-4t^2)}\Big) - 2(2t^2-1)^2 - (n-2)(\sqrt{2}t+1)^2 \leqslant 0.$$

We notice that $(4t^2-1)(n-4t^2) \leqslant (n-2)$ for $t \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$ and $n \geqslant 3$. Thus, it is enough to show that

$$\big(2(n-3)t^2 + n - 1\big) - 2(2t^2-1)^2 - (n-2)(\sqrt{2}t+1)^2 \leqslant 0,$$

which becomes trivial after simplification. □

We can now apply Theorem 3 (as we did with equation (4)) to $(n-m)$-dimensional faces of Q_n (for $m = 0, 1, \ldots, n-3$) and Theorem 1 to the 2-dimensional face to give us the general result:

Theorem 4. *Let $m = 0, 1, \ldots, n-2$. Then for a line ℓ that is distance $t \in \left[\frac{\sqrt{m}}{2}, \frac{\sqrt{m+1}}{2}\right]$ away from the origin, we have that*

$$\max_{\ell \subset \mathbb{R}^n, d(\ell,0)=t} |Q_n \cap \ell| = \frac{[n-m]\sqrt{n-m-1}}{\sqrt{(n-m-1)[n-(2t)^2]} + 2\sqrt{t^2 - \frac{m}{4}}}$$

and

$$\max_{\ell\subset\mathbb{R}^n, d(\ell,0)=t} |Q_n\cap\ell| = \begin{cases} \sqrt{t^2-\frac{n-2}{4}}-\sqrt{t^2-\frac{n-1}{4}}, & t\in(\frac{\sqrt{n-1}}{2}, \frac{1}{4}\sqrt{\frac{8n-7}{2}}], \\ \sqrt{2}-2\sqrt{t^2-\frac{n-2}{4}}, & t\in(\frac{1}{4}\sqrt{\frac{8n-7}{2}}, \frac{\sqrt{n}}{2}] \end{cases}.$$

Proof. In case $0\leqslant t\leqslant\frac{1}{2}$, Theorem 3 gives

$$\max_{\ell\subset\mathbb{R}^n, d(\ell,0)=t} |Q_n\cap\ell| = \frac{n\sqrt{n-1}}{\sqrt{(n-1)(n-4t^2)}+2t}.$$

Now we consider the case $t>1/2$. Let $t\in[\frac{\sqrt{m}}{2}, \frac{\sqrt{m+1}}{2}]$ for $m=1,\dots,n-1$. Then the second part of Theorem 3 says that a maximal ℓ should be confined to a face F of Q_n, and moreover there exist two edges E_1, E_2 of Q_n such that the line ℓ contains two points on each E_1, E_2. Choose a face F in the following way:

(1) If E_1, E_2 meet to a vertex, then consider the 2-dimensional face F of Q_n containing E_1, E_2

(2) Otherwise, then consider the smallest (in dimension) face of Q_n containing E_1, E_2 whose center is the midpoint of two vertices in E_1, E_2.

The dimension of F is equal to $n-m$, and the center of F is $\sqrt{m}/2$ far away from the origin. Thus, the line segment $Q_n\cap\ell$ is contained in F and it is $r=\sqrt{t^2-m/4}$ away from the center of F. In addition the line ℓ passes through a $(n-m)$-dimensional cube, $Q_n\cap F$, whose center is the same as F.

Next, we use Theorem 3 for the $(n-m)$-dimensional cube and the line which is $r=\sqrt{t^2-m/4}\in[0,1/2]$ away from the center of the cube. Then

$$\begin{aligned}\max_{\ell\subset\mathbb{R}^n, d(\ell,0)=t} |Q_n\cap\ell| &= \max_{\ell'\subset\mathbb{R}^{n-m}, d(\ell',0)=r} |Q_{n-m}\cap\ell'| \\ &= \frac{(n-m)\sqrt{(n-m)-1}}{\sqrt{(n-m-1)(n-m-4r^2)}+2r} \\ &= \frac{(n-m)\sqrt{n-m-1}}{\sqrt{(n-m-1)(n-4t^2)}+2\sqrt{t^2-m/4}}.\end{aligned}$$

For the case $m=n-1$, i.e., $t\in[\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$, apply Theorem 1 with $r=\sqrt{t^2-(n-2)/4}$. ☐

3. HYPERPLANE AND SLAB SECTIONS AT DISTANCE $t>\frac{\sqrt{n-1}}{2}$

In this section we present a partial solution to the problem of finding extremal sections of Q_n by affine hyperplanes tangent to $t\mathbb{S}^{n-1}$ and slabs of width $2t$. Let $H(\xi,t)=\{x\in\mathbb{R}^n : x\cdot\xi=t\}$ be a hyperplane with a normal vector $\xi\in\mathbb{S}^{n-1}$ at the distance $t>0$ from the origin.

Theorem 5. *Fix* $t\in(\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$. *Then, for* $n\geqslant 3$,

$$|Q_n\cap H(\xi,t)|\leq\frac{n^{\frac{n}{2}}}{(n-1)!}\left(\frac{\sqrt{n}}{2}-t\right)^{n-1}, \text{ for all } \xi\in\mathbb{S}^{n-1}$$

with equality occurring when $\xi=\left(\pm\frac{1}{\sqrt{n}},\dots,\pm\frac{1}{\sqrt{n}}\right)$.

Remark 2. *It is interesting to compare Theorems 1 and 5 and to see the difference between maximal sections of Q_2 and Q_n, $n \geqslant 3$.*

Proof. Fix $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$. The main idea of the proof is to notice the geometric implications of this restriction. Indeed, $\frac{\sqrt{n}}{2}$ is the distance from the origin to a vertex of Q_n and $\frac{\sqrt{n-1}}{2}$ is the distance from the origin to an edge of Q_n. Thus, our hyperplane $H(\xi, t)$ cannot separate an entire edge from the origin, nor can it separate two vertices from the origin.

Each vertex is the intersection of precisely n edges, so $H(\xi, t)$ cannot intersect more than n edges. Furthermore, since $H(\xi, t)$ cannot be contained in an $(n-1)$-dimensional face, then it must intersect exactly n edges and separate one vertex from the rest of Q_n (provided that it intersects Q_n at all). By symmetry, we may assume that the separated vertex is $\mathbf{v} = (\frac{1}{2}, \dots, \frac{1}{2})$.

Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^n$ be the points of intersection of $H(\xi, t)$ with each respective edge extending from $\mathbf{v}$, such that $\mathbf{a}_i$ belongs to the edge parallel to coordinate vector $\mathbf{e}_i$, i.e. all coordinates of $\mathbf{a}_i$ are $\frac{1}{2}$, except for the i-th coordinate, which is $\frac{1}{2} - a_i$, where $a_i = |\mathbf{a}_i - \mathbf{v}| > 0$. Moreover, $a_i < \frac{1}{2}$ because $t\mathbb{S}^{n-1}$ intersects all edges of Q_n.

We notice that $H(\xi, t)$ cuts a simplex $S(\xi, t)$ from Q_n, with an apex $\mathbf{v}$ and a base which is the convex hull of $\{\mathbf{a}_i\}_{i=1}^n$. We also note that $\mathbf{v}$ is an "orthogonal corner" of $S(\xi, t)$, i.e. all facets containing $\mathbf{v}$ are pairwise orthogonal.

Let F be the face of $S(\xi, t)$ opposite from $\mathbf{v}$ and let F_i, for $i = 1, 2, \dots, n$, be the face that is opposite from the vertex $\mathbf{a}_i$. We can compute $|Q_n \cap H(\xi, t)| = |F|$ as a function of the a_i (instead of ξ and t). First note that

$$|F_i| = \frac{1}{(n-1)!} \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j.$$

Using a classical n-dimensional analog of the Pythagorean theorem (see, for example, [Al]) we claim that $|F|^2 = \sum_{i=1}^n |F_i|^2$ and thus,

$$|H(\xi, t) \cap Q_n|^2 = |F|^2 = \left(\frac{1}{(n-1)!}\right)^2 \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j^2.$$

We want to maximize $|F|$ subject to the constraint that the unique hyperplane containing F is at distance t from the origin. In order to do this, we should express the distance of the (unique) hyperplane containing $\mathbf{a}_1, \dots, \mathbf{a}_n$ from the origin in terms of $a_1, \dots, a_n$. This distance can be expressed as $\mathbf{a}_1 \cdot \xi$, so it suffices to find ξ in terms of $\{a_i\}_{i=1}^n$.

The normal vector ξ can be characterized as the unique vector satisfying $|\xi| = 1$ and $\xi \cdot \mathbf{a}_i = t$ for $i = 1, \dots, n$. Calculation shows:

$$\xi = \frac{(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n})}{\sqrt{\sum_{i=1}^n \frac{1}{a_i^2}}}.$$

Thus

$$\mathbf{a}_1 \cdot \xi = \frac{\left(\sum_{i=1}^n \frac{1}{a_i}\right) - 2}{2\sqrt{\sum_{i=1}^n \frac{1}{a_i^2}}}.$$

We will now apply the method of Lagrange multipliers, to optimize the function

$$f(a_1,\dots,a_n) := \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n} a_j^2 = \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2},$$

with the constraint

$$g(a_1,\dots,a_n) := \sum_{i=1}^{n}\frac{1}{a_i} - 2t\sqrt{\sum_{i=1}^{n}\frac{1}{a_i^2}} = 2. \tag{5}$$

Function f is differentiable everywhere on $\mathbb{R}^n$ and g is differentiable everywhere on $\mathbb{R}^n \setminus \{0\}$, moreover, ∇g is non-zero. From now on, we will consider f and g as functions from $(0,\frac{1}{2})^n$ to $\mathbb{R}$. Under these conditions, every global extremum $\mathbf{c}$ of f with constraint $g = 2$ satisfies $(\nabla f - \lambda\nabla g)(\mathbf{c}) = 0$ for some $\lambda \in \mathbb{R}$.

Let $F_\lambda := f - \lambda g$. Then,

$$\frac{\partial F_\lambda}{\partial a_k} = 2a_k \prod_{\substack{j=1\\ j\neq k}}^{n} a_j^2 \sum_{i=1}^{n}\frac{1}{a_i^2} - \frac{2}{a_k^3}\prod_{j=1}^{n} a_j^2 - \lambda\left(-\frac{1}{a_k^2} + 2t\frac{\frac{1}{a_k^3}}{\sqrt{\sum_{i=1}^{n}\frac{1}{a_i^2}}}\right).$$

To find possible extremal values we need to solve $\frac{\partial F_\lambda}{\partial a_k} = 0$, for all $k = 1,\dots,n$, which is:

$$2a_k \prod_{\substack{j=1\\ j\neq k}}^{n} a_j^2 \sum_{i=1}^{n}\frac{1}{a_i^2} - \frac{2}{a_k^3}\prod_{j=1}^{n} a_j^2 - \lambda\left(-\frac{1}{a_k^2} + 2t\frac{\frac{1}{a_k^3}}{\sqrt{\sum_{i=1}^{n}\frac{1}{a_i^2}}}\right) = 0.$$

Multiplying the above equation by a_k, we get

$$2\prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n}\frac{1}{a_i^2} - \frac{2}{a_k^2}\prod_{j=1}^{n} a_j^2 - \lambda\left(-\frac{1}{a_k} + 2t\frac{\frac{1}{a_k^2}}{\sqrt{\sum_{i=1}^{n}\frac{1}{a_i^2}}}\right) = 0. \tag{6}$$

To find λ, we will sum up equations (6) for $k = 1,\dots,n$:

$$2(n-1)\prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n}\frac{1}{a_i^2} - \lambda\left(-\sum_{i=1}^{n}\frac{1}{a_k} + 2t\sqrt{\sum_{i=1}^{n}\frac{1}{a_i^2}}\right) = 0.$$

Thus,

$$\lambda = \frac{2(n-1)\prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n}\frac{1}{a_i^2}}{2t\sqrt{\sum_{i=1}^{n}\frac{1}{a_i^2}} - \sum_{i=1}^{n}\frac{1}{a_k}} = -(n-1)\prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n}\frac{1}{a_i^2}, \tag{7}$$

with the last equality following from (5). Next, fix two different indices $k, m \in [1,\dots,n]$. We subtract from (6) multiplied by $1/a_m^2$ the corresponding equation for m multiplied by $1/a_k^2$ to get

$$\left(2\prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n}\frac{1}{a_i^2}\right)\left(\frac{1}{a_m^2} - \frac{1}{a_k^2}\right) + \lambda\left(\frac{1}{a_k a_m^2} - \frac{1}{a_m a_k^2}\right) = 0.$$

Thus, we get that the possible extremal point either must satisfy $\frac{1}{a_m} - \frac{1}{a_k} = 0$ or must satisfy

$$\left(2\prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2}\right)\left(\frac{1}{a_m}+\frac{1}{a_k}\right)+\frac{\lambda}{a_k a_m}=0,$$

which is equivalent to

$$\left(2\prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2}\right)(a_k+a_m)=-\lambda.$$

Finally, using (7) we get

$$a_k + a_m = \frac{n-1}{2}.$$

The above equality is only possible for $n=2$ (indeed, $a_i \in [0, \frac{1}{2}]$). Thus, the only critical point is one satisfying $a_1 = a_2 = \cdots = a_n$. This yields that $\xi = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$.

We must now find whether this critical point is actually a global maximum subject to $g=2$. To do so we will partition $[0, \frac{1}{2}]^n$ into two pieces: one compact set C containing our critical point, and a set $U = [0, \frac{1}{2}]^n \setminus C$ on which f is small when constrained by $g=2$. As f is continuos, $f|_C$ attains a (constrained) maximum on C subject to $g=2$. If f subject to $g=2$ is uniformly bounded on U by this same (constrained) maximum, then f reaches a global (constrained) maximum when $a_1 = a_2 = \cdots = a_n$.

We consider set $C = [\delta, \frac{1}{2} - \delta]^n$, for some small $\delta > 0$. Fixing $t > (n-1)/2$ and considering only $(a_1..., a_n)$ satisfying (5), it is easy to select δ such that $a_i \leqslant 1/2 - \delta$ for all $i = 1, \ldots, n$. Thus if $(a_1, \cdots, a_n) \in U$, then at least one of the a_i must be less than δ, which gives:

$$f(a_1, \ldots, a_n) \leqslant \delta \frac{n}{4^n}.$$

Thus one can select $\delta > 0$ such that $f(a_1, \ldots, a_n) \leqslant f(\xi)$ for $\xi = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$, which would guarantee that $f(\xi)$ is the global maximum.

By plugging in $a := a_1 = \cdots = a_n$ into (5), we find that

$$a = \sqrt{n}(\frac{\sqrt{n}}{2} - t),$$

which yields that for $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$,

$$\max_{\xi \in \mathbb{S}^{n-1}} |Q_n \cap H(\xi, t)| = \frac{1}{(n-1)!}\sqrt{\sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} a_j^2} = \frac{\sqrt{n}}{(n-1)!} a^{(n-1)} = \frac{n^{\frac{n}{2}}}{(n-1)!}\left(\frac{\sqrt{n}}{2} - t\right)^{n-1}.$$

□

The ideas used in the proof of Theorem 5 can also be used to give a characterization of minimal volume sections of Q_n by slabs with large width:

Theorem 6. *Fix $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$. Then,*

$$|Q_n \cap \{x \in \mathbb{R}^n : |x \cdot \xi| \leqslant t\}| \geq 1 - \frac{2n^{\frac{n}{2}}}{n!}\left(\frac{\sqrt{n}}{2} - t\right)^n, \text{ for all } \xi \in \mathbb{S}^{n-1}$$

with equality occurring when $\xi = \left(\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}}\right)$.

Proof. The proof for two-dimensional case was provided in [BK]. Thus, we will concentrate on $n \geqslant 3$. Fix $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$. It is clear that the minimal slab must not contain the whole Q_n, and thus, must cut two congruent pyramids from Q_n (see the discussion in the beginning of the proof of Theorem 5). By the symmetry, we may assume that one of the pyramids have $\mathbf{v} = (\frac{1}{2}, \dots, \frac{1}{2})$ as its apex. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^n$ be as define in the proof of Theorem 4 and thus,

$$|Q_n \cap \{x \in \mathbb{R}^n : |x \cdot \xi| \leqslant t\}| = 1 - \frac{2}{n!} \prod_{i=1}^n a_i.$$

So, we may again apply the method of Lagrange multipliers, to optimize the function

$$f(a_1, \dots, a_n) = \prod_{i=1}^n a_i, \text{ with the constraint } \sum_{i=1}^n \frac{1}{a_i} - 2t \sqrt{\sum_{i=1}^n \frac{1}{a_i^2}} = 2. \tag{8}$$

Similarly, as in the proof of Theorem 5, we consider $(a_1, \dots, a_n) \in (0, \frac{1}{2})^n$ and set

$$F_\lambda(a_1, \dots, a_n) = \prod_{i=1}^n a_i - \lambda \left(\sum_{i=1}^n \frac{1}{a_i} - 2 - 2t \sqrt{\sum_{i=1}^n \frac{1}{a_i^2}} \right),$$

then

$$\frac{\partial F_\lambda}{\partial a_k} = \prod_{\substack{i=1 \\ i \neq k}}^n a_i - \lambda \left(-\frac{1}{a_k^2} + 2t \frac{\frac{1}{a_k^3}}{\sqrt{\sum_{i=1}^n \frac{1}{a_i^2}}} \right).$$

To find possible extremal values we need to solve $\frac{\partial F_\lambda}{\partial a_k} = 0$, for all $k = 1, \dots, n$, which is:

$$\prod_{i=1}^n a_i - \lambda \left(-\frac{1}{a_k} + 2t \frac{\frac{1}{a_k^2}}{\sqrt{\sum_{i=1}^n \frac{1}{a_i^2}}} \right) = 0. \tag{9}$$

To find λ, we will sum up equations (9) for $k = 1, \dots, n$ and use (8) to get:

$$\lambda = -\frac{n}{2} \prod_{j=1}^n a_j. \tag{10}$$

Next, we fix two different indices $k, m \in [1, \dots, n]$. Then, we subtract from (9) multiplied by $1/a_m^2$ the corresponding equation for m multiplied by $1/a_k^2$ we get

$$\left(\prod_{j=1}^n a_j \right) \left(\frac{1}{a_m^2} - \frac{1}{a_k^2} \right) + \lambda \left(\frac{1}{a_k a_m^2} - \frac{1}{a_m a_k^2} \right) = 0.$$

Thus, again, we get that the possible extremal point must satisfy either $\frac{1}{a_m} - \frac{1}{a_k} = 0$ or

$$\left(\prod_{j=1}^n a_j \right) \left(\frac{1}{a_m} + \frac{1}{a_k} \right) + \frac{\lambda}{a_k a_m} = 0.$$

The above again gives $a_k + a_m = \frac{n-1}{2}$, which is only possible for $n = 2$. Thus, for $n \geqslant 3$ the only possible critical point is $a_1 = a_2 = \dots = a_n$. By arguments similar to those used in

Theorem 5 (working with set C and δ), we may show f achieves a (constrained) maximum at $a_1 = a_2 = \cdots = a_n$.

This yields the maximal $\xi = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$ and $a = \sqrt{n}(\frac{\sqrt{n}}{2} - t)$. This gives, for $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$,

$$|Q_n \cap \{x \in \mathbb{R}^n : |x \cdot \xi| \leqslant t\}| \geq 1 - \frac{2n^{\frac{n}{2}}}{n!} \left(\frac{\sqrt{n}}{2} - t \right)^n .$$

$\square$

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