

A remark on p -summing norms of operators

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ABSTRACT. In this paper we improve a result of W. B. Johnson and G. Schechtman by proving that the p -summing norm of any operator with n -dimensional domain can be well-approximated using $C(p)n \log n (\log \log n)^2$ vectors if $1 < p < 2$, and using $C(p)n^{p/2} \log n$ if $2 < p < \infty$.

1. p -summing norms

Throughout this paper we will follow notations of N. Tomczak-Jaegermann [To].

DEFINITION 1.1. Let X and Y be Banach spaces. An operator $u : X \rightarrow Y$ is called **p -summing** if there exists a constant c such that for all finite sequences $\{x_j\}$ in X one has

$$\left(\sum_j \|ux_j\|^p \right)^{1/p} \leq c \sup_{x^* \in X^* \text{ and } \|x^*\| \leq 1} \left(\sum_j |(x_j, x^*)|^p \right)^{1/p}.$$

The infimum of constants c satisfying this inequality is denoted by $\pi_p(u)$ and is called the p -summing norm of u .

Here we would like to discuss the following natural question: Given an operator u with n -dimensional domain, how many vectors do we need to approximate the p -summing norm of u ?

To present a more precise version of this question we will first give the following definition:

DEFINITION 1.2. For positive integer n , let $\pi_p^{(n)}(u)$ is the smallest constant c such that for arbitrary vectors $x_1, \dots, x_n \in X$ one has

$$\left(\sum_{j=1}^n \|ux_j\|^p \right)^{1/p} \leq c \sup_{x^* \in X^* \text{ and } \|x^*\| \leq 1} \left(\sum_{j=1}^n |(x_j, x^*)|^p \right)^{1/p}.$$

One can see that

$$\|u\| = \pi_p^{(1)}(u) \leq \pi_p^{(2)}(u) \leq \dots \leq \pi_p^{(n)}(u) \leq \pi_p(u),$$

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and

$$\lim_{n \rightarrow \infty} \pi_p^{(n)}(u) = \pi_p(u).$$

Now we are ready to state a precise version of the question above: Given a operator u with n -dimensional domain what is the smallest number N such that $\pi_p(u) \leq C\pi_p^{(N)}(u)$.

The first case to study is $p = 2$. In this case a useful result of N. Tomczak-Jaegermann (see [T**o**], page 143) states that if u is 2-summing of rank n than $\pi_2(u) \leq \sqrt{2}\pi_2^{(n)}(u)$.

For $p = 1$, S. Szarek [S**z**] proved that $\pi_1(u) \leq C\pi_1^{(Cn \log n)}(u)$. Finally W. B. Johnson and G. Schechtman proved in [J-S], that $\pi_p(u) \leq C\pi_p^{(Cn \log^3 n)}(u)$ for $1 < p < 2$ and $\pi_p(u) \leq C\pi_p^{(Cn^{p/2} \log^3 n)}(u)$, for $2 < p < \infty$.

Here we would like to improve a result of W. B. Johnson and G. Schechtman by reducing the power of $\log n$. We will follow an approach of [J-S] but we will use different methods for approximation of the expectation (or tails) of a given random process. Our goal is to prove the following theorem:

THEOREM 1.1. *Suppose that $\dim(X) = n$, $u : X \rightarrow Y$ is a linear operator and $\varepsilon > 0$. Then*

$$\pi_p(u) \leq (1 + \varepsilon)\pi_p^{(N)},$$

as long as

- (i): $1 < p < 2$ and $N \geq C(p)\varepsilon^{-2}n(\log n)(\log \log(\varepsilon^{-2}n) + \log \varepsilon^{-1})^2$,
- (ii): $2 < p < \infty$ and $N \geq C(p)\varepsilon^{-5}n^{\frac{p}{2}} \log(\varepsilon^{-5}n)$.

Given a linear operator $u : X \rightarrow Y$ of finite rank, $1 < q < \infty$ and positive integers n, k define

$$\nu_q^{(n,k)}(u) = \inf \left\{ \sum_{i=1}^k \nu_q^{(n)}(u_i) : u = \sum_{i=1}^k u_i \right\},$$

where $\nu_q^{(n)}(u)$ is the infimum of

$$\|A\| \|w\| \|B\|$$

where $u = BwA$ and

$$A : X \rightarrow \ell_\infty^n, w : \ell_\infty^n \rightarrow \ell_q^n \text{ diagonal}, B : \ell_q^n \rightarrow Y.$$

The next theorem (theorem 24.2 [T**o**]) shows a connection between $\pi_p^n(u)$ and $\nu_q^{(n,k)}(u)$:

THEOREM 1.2. *Let $\frac{1}{p} + \frac{1}{q} = 1$ and*

$$\hat{\nu}_q^{(n)}(u) = \lim_{k \rightarrow \infty} \nu_q^{(n,k)}(u),$$

then the ideal norms $\pi_p^{(n)}$ and $\hat{\nu}_q^{(n)}$ are in trace duality.

The next theorem is a key step in the proof of theorem 1.1. After proving it, we will finish our proof of theorem 1.1 using an iterational procedure, and theorem 1.2.

THEOREM 1.3. *Let $n \leq M$ be positive integers; $u : X \rightarrow Y$ a linear operator with X finite dimensional and $\dim(Y) \leq n$. Then, (for $\frac{1}{p} + \frac{1}{q} = 1$),*

(i): For $1 < p < 2$,

$$\nu_q^{(\frac{7}{8}M, 2)}(u) \leq \left(1 + C_p \left(\frac{n}{M} \log n \left[\log \frac{M}{n} + \log \log M\right]^2\right)^{\frac{1}{2}}\right) \nu_q^{(M, 1)}(u).$$

(ii): For $2 < p < \infty$,

$$\nu_q^{(\frac{7}{8}M, 2)}(u) \leq \left(1 + c(p) \left(\frac{n^{p/2}}{M} \log M\right)^{1/5}\right) \nu_q^{(M, 1)}(u).$$

2. Probabilistic lemma

Before proving theorem 1.3 we need to prove the following probabilistic lemma, which will improve the result of proposition 2.1, [J-S].

The assumptions of this lemma is the ‘‘Lewis lemma’’ (see [L], or [S-Z]): in [S-Z] it was shown that for any n -dimensional subspace X of $L_p(\Omega, \mu)$, $0 < p < \infty$, one can find a probability measure τ on Ω and a subspace \tilde{X} of $L_p(\Omega, \tau)$ isometric to X which admits a basis h_1, \dots, h_n orthonormal in $L_2(\Omega, \tau)$, such that

$$\sum_{i=1}^n h_i^2 \equiv n.$$

In addition we note that if X is a subspace of ℓ_p^M and τ is the probability measure on $\{1, \dots, M\}$ given by the theorem above then, as observed in [J-S], we can split each atom of τ of mass large than $4/M$ into pieces each of size between $2/M$ and $4/M$. This will enlarge the number of atoms by at most $M/2$. The new measure λ is such that

$$\lambda\{i\} \leq 4/M \text{ for all } i,$$

and it is supported on $\bar{K} = \{1, \dots, K\}$, where $K \leq 3M/2$. Finally, \tilde{X} is still isometric (we denote this isometry by J) to a subspace $J\tilde{X}$ of $L_p(\bar{K}, \lambda)$, and $J\tilde{X}$ admits an orthonormal basis h_1, \dots, h_n whose sum of squares is a constant (clearly, X is also isometric to $J\tilde{X}$).

LEMMA 2.1. *Let X be an n dimensional subspace of $L_p(\bar{K}, \lambda)$ with $2 < p < \infty$, and assume that X has characteristics from the Lewis lemma (plus the remark after it, concerning the splitting of big atoms). Let B_p be the unit ball of X and $\{\varepsilon_j\}_{j=1}^K$ be independent random variables taking the values $+1$ or -1 with equal probability, then*

$$\mathbb{P} \left\{ \sup_{x \in B_p} \left| \sum_{j=1}^K \lambda\{j\} \varepsilon_j |x(j)|^p \right| > c(p) \left(\frac{n^{p/2}}{K} \log K \right)^{1/5} \right\} \leq \frac{1}{2}.$$

Proof: To prove this lemma we will use a method of proof due to J. Bourgain, J. Lindenstrauss and V. Milman [B-L-M].

Fix $0 < t < 1/2$ to be chosen later ($t = c(p) \left(\frac{n^{p/2}}{K} \log K \right)^{1/5}$).

Let \mathcal{F} be a $\frac{1}{2}t$ -net (with respect of the metric $d(x, y) = \sum \lambda\{j\} ||x_i|^p - |y_i|^p|$) on the boundary of B_p then,

$$\begin{aligned} d(x, y) &= \sum \lambda\{j\} ||x_i|^p - |y_i|^p| \leq p \sum \lambda\{j\} \max\{|x_i|, |y_i|\}^{p-1} |x_i - y_i| \leq \\ &p \left(\sum \lambda\{j\} \max\{|x_i|, |y_i|\}^p \right)^{(p-1)/p} \left(\sum \lambda\{j\} |x_i - y_i|^p \right)^{1/p} \leq c_p ||x - y||_p, \end{aligned}$$

so by a standard volume argument, we get that $\log |\mathcal{F}| \leq c_p n \log t^{-1}$.

It is easy to see that:

$$\mathbb{P} \left\{ \sup_{x \in B_p} \left| \sum_{j=1}^K \lambda\{j\} \varepsilon_j |x(j)|^p \right| > \frac{1}{2} t \right\} \leq \mathbb{P} \left\{ \sup_{x \in \mathcal{F}} \left| \sum_{j=1}^K \lambda\{j\} \varepsilon_j |x(j)|^p \right| > t \right\}.$$

Using the Lewis lemma (see [**J-S**], Proposition 2.1) we get that $\|x\|_\infty \leq n^{1/2} \|x\|_p$, for $x \in X$, so that the l_∞ -diameter of B_p is less or equal to $2n^{1/2}$.

For $k = 1, 2, \dots, l = \lceil \log_{(1+t)} n^{1/2} \rceil + 1$ let $A_k \subset B_p$ be such that

$$B_p \subset \bigcup_{g \in A_k} \left(g + \frac{t(1+t)^k}{3} B_\infty \right).$$

X has characteristics from the Lewis lemma, so applying Proposition 2.1 from [**J-S**] [$\log N(B_p, B_\infty, t) \leq c(p)n \log K t^{-2}$] we get:

$$\log |A_k| \leq c(p)n \log K t^{-2} (1+t)^{-2k}.$$

For every $x \in \mathcal{F}$ and $1 \leq k \leq l$ let $f^{k,x} \in A_k$ satisfy $\|x - f^{k,x}\|_\infty \leq t(1+t)^k/3$. Put

$$C_{k,x} = \{i : |f_i^{k,x}| \geq (1+t)^{k-1}\},$$

$$D_{k,x} = C_{k,x} \setminus \bigcup_{h>k} C_{h,x}, \quad D_{0,x} = \{1, \dots, K\} \setminus \bigcup_{k=1}^l C_{k,x},$$

and

$$\hat{x} = x I_{D_{0,x}} + \sum_{k=1}^l (1+t)^k I_{D_{k,x}}.$$

Note that if $i \in C_{k,x}$, $k \geq 1$, then

$$|x_i| \geq (1+t)^{k-1} - t(1+t)^k/3 \geq (1+t)^{k-2},$$

while if $i \notin C_{k,x}$

$$|x_i| \leq (1+t)^{k-1} + t(1+t)^k/3 \leq (1+t)^{k+1}.$$

Hence for every $i \in D_{k,x}$, $k \geq 1$,

$$(1+t)^{k-2} \leq |x_i| \leq (1+t)^{k+2},$$

while for $i \in D_{0,x}$, $|x_i| \leq (1+t)^2$. It follows that

$$(1+t)^{-2} \leq \frac{|\hat{x}_i|}{|x_i|} \leq (1+t)^2,$$

then

$$1 - 4pt \leq \frac{|\hat{x}_i|^p}{|x_i|^p} \leq 1 + 4pt.$$

Then

$$\left| \sum_{j=1}^K \lambda\{j\} \varepsilon_j |\hat{x}(j)|^p - \sum_{j=1}^K \lambda\{j\} \varepsilon_j |x(j)|^p \right| \leq \sum_{j=1}^K \lambda\{j\} \left| |\hat{x}(j)|^p - |x(j)|^p \right| \leq c_p t.$$

So we may consider \hat{x} instead of x .

To prove our lemma we first estimate the probability of the following event:

$$\left| \sum_{j=1}^K \lambda\{j\} \varepsilon_j |\hat{x}(j)|^p I_{D_{0,x}} \right| \leq t_0, \quad x \in \mathcal{F},$$

$$\left| \sum_{j=1}^K \lambda\{j\} \varepsilon_j (1+t)^{pk} I_{D_{k,x}} \right| \leq t_k, \quad 1 \leq k \leq l, x \in \mathcal{F},$$

where

$$\sum t_k \leq t.$$

Let B_k $1 \leq k \leq l$ be the collection of all sets of form $D_{k,x}$. It follows from the definition that

$$\log |B_k| \leq \sum_{h=k}^l \log |A_h| \leq c_p n \log K t^{-3} (1+t)^{-2k}.$$

Now we can apply Bernstein's inequality (see, for example, [B], page 39), which states that for given $\{a_i\}_{i=1}^n \in \mathbb{R}^n$, with $\sum_{i=1}^n a_i^2 = 1$ we get

$$\mathbb{P}\left(\left|\sum a_i \varepsilon_i\right| > u\right) \leq 2 \exp\left(\frac{-u^2}{2}\right).$$

So that if

$$a_j = \frac{\lambda\{j\} |\hat{x}(j)|^p}{\sum_{j \in D_{0,x}} \lambda^2\{j\} |\hat{x}(j)|^{2p}}, \quad u = \frac{t_0}{\sum_{j \in D_{0,x}} \lambda^2\{j\} |\hat{x}(j)|^{2p}},$$

then

$$\mathbb{P}\left(\left|\sum_{j=1}^K \lambda\{j\} \varepsilon_j |\hat{x}(j)|^p I_{D_{0,x}}\right| > t_0\right) \leq 2 \exp\left(\frac{-t_0^2}{2 \sum_{j \in D_{0,x}} \lambda^2\{j\} |\hat{x}(j)|^{2p}}\right).$$

Applying $\sum_{j \in D_{0,x}} \lambda\{j\} |\hat{x}(j)|^p \leq 2$, $\lambda\{j\} |\hat{x}(j)|^p \leq \frac{2}{K} (1+t)^{2p}$, we get

$$\mathbb{P}\left(\left|\sum_{j=1}^K \lambda\{j\} \varepsilon_j |\hat{x}(j)|^p I_{D_{0,x}}\right| > t_0\right) \leq 2 \exp(-c_p t_0^2 K (1+t)^{-2p}).$$

In the same manner we get:

$$\mathbb{P}\left(\left|\sum_{j=1}^K \lambda\{j\} \varepsilon_j |\hat{x}(j)|^p I_{D_{k,x}}\right| > t_k\right) < 2 \exp(-c_p t_k^2 K (1+t)^{-pk}).$$

So, in order to prove our lemma is enough to estimate

$$2|\mathcal{F}| \exp(-c_p t_0^2 K) + 2 \sum_{k=1}^l |B_k| \exp(-c_p t_k^2 K (1+t)^{-pk}) =$$

$$2 \exp(4n \log t^{-1} - c_p t_0^2 K)$$

$$+ c_p \sum_{k=1}^l \exp(c_p n \log K t^{-3} (1+t)^{-2k} - c_p t_k^2 K (1+t)^{-pk}) =$$

Let $t_o = c_1(p)t$, $t_l = c_2(p)t$, $t_k = c_2(p)(1+t)^{(2-p)(l-k)/2}t$, then

$$\begin{aligned}
&= 2 \exp(4n \log t^{-1} - c_p t^2 K) \\
&\quad + c_p \sum_{k=1}^l \exp(c_p n \log K t^{-3} (1+t)^{-2k} - c_p K (1+t)^{(2-p)(l-k)-pk} t^2) = \\
&= 2 \exp(4n \log t^{-1} - c_p t^2 K) \\
&\quad + c_p \sum_{k=1}^l \exp(c_p t^{-3} (1+t)^{-2k} (n \log K - c_p K (1+t)^{2l-pl} t^5)) = \\
&= 2 \exp(4n \log t^{-1} - c_p t^2 K) + c_p \sum_{k=1}^l \exp(c_p (1+t)^{-2k} n (\log K - c_p K n^{-p/2} t^5)).
\end{aligned}$$

To conclude the prove of our lemma we take $t = c_p \left(\frac{n^{p/2}}{K} \log K \right)^{1/5}$ (i.e. such that $\log K - c_p K n^{-p/2} t^5 < -c'_p$).

□

The next lemma shows that in some applications of lemma 2.1, we may “omit” requirements for “Lewis lemma” characteristics.

LEMMA 2.2. *Let X be an n -dimensional subspace of $L_p(\bar{M}, \mu)$ consider $JX \subset L_p(\bar{K}, \mu')$, where J is the isometry defined by the splitting of big atoms. Then there is a partition $K_1 \cup K_2$ of \bar{K} into two sets of cardinality at most $\frac{7}{8}M$ such that for each $x \in JX$ and $j = 1, 2$:*

(i):

$$\|1_{K_j} x\|_{L_p(\bar{K}, \mu')}^p \leq \left(\frac{1}{2} + C_p \left(\frac{n}{M} \log n \left[\log \frac{M}{n} + \log \log M \right]^2 \right)^{\frac{1}{2}} \right) \|x\|_{L_p(\bar{M}, \mu)}^p,$$

when $1 < p < 2$;

(ii):

$$\|1_{K_j} x\|_{L_p(\bar{K}, \mu')}^p \leq \left(\frac{1}{2} + c(p) \left(\frac{n^{p/2}}{M} \log M \right)^{1/5} \right) \|x\|_{L_p(\bar{K}, \mu)}^p,$$

for $2 < p < \infty$.

Proof: To prove (i) we use a result of M. Talagrand from his paper on embedding n -dimensional subspaces of L_p into L_p^N with N not too large (see proof of Proposition 2.3 from [T]) and get that if $Z \in L_p(\bar{M}, \tau)$ is a isometric copy of X after a Lewis change of density, and if $JZ \subset L_p(\bar{K}, \lambda)$ then

$$(1) \quad \sup_{z \in B_p(JZ)} \left| \sum_{i=1}^K \lambda\{i\} \varepsilon_i |z(i)|^p \right| \leq C_p \left(\frac{n}{K} \log n \left[\log \frac{K}{n} + \log \log K \right]^2 \right)^{\frac{1}{2}}$$

for most choices of signs $\varepsilon_i = \pm 1$.

One can see that for a fixed J the left hand side is invariant under a change of density f , i.e. we can replace the subspace Y of $L_p(\bar{K}, \lambda)$ with its image under

the natural isometry from $L_p(\bar{K}, \lambda)$ to $L_p(\bar{K}, \mu)$ defined by $Tx = x/f^{\frac{1}{p}}$. And using that $M < K < \frac{3}{2}M$ we get

$$\sup_{x \in B_p(JX)} \left| \sum_{i=1}^K \mu'\{i\} \varepsilon_i |z(i)|^p \right| \leq C_p \left(\frac{n}{M} \log n \left[\log \frac{M}{n} + \log \log M \right]^2 \right)^{\frac{1}{2}}$$

for most choices of signs $\varepsilon_i = \pm 1$. Since also for most choices of signs the difference between the number of plus signs and minus signs is less than $K/8$, (i) follow. (ii) follow similarly, using lemma 2.1. \square

3. Proof of the main result

Proof of Theorem 1.3: For some probability measure μ on $\bar{M} = \{1 \dots M\}$, we consider $A : Y^* \rightarrow L_p(\bar{M}, \mu)$, and $B : L_1(\bar{M}, \mu) \rightarrow X^*$ so that $\|A\| \|B\| = \nu_q^{(M)}(u)$ and $u^* = B i_{p,1} A$, where $i_{p,1}$ is the formal identity mapping from $L_p(\bar{M}, \mu)$ onto $L_1(\bar{M}, \mu)$.

Next, using lemma 2.2, we get a partition $K_1 \cup K_2 = \bar{K}$ into two sets of cardinality at most $\frac{7}{8}M$ such that

$$\|1_{K_j} J A y\|_{L_p(\bar{K}, \mu')}^p \leq \left(\frac{1}{2} + c(p) \left(\frac{n^{p/2}}{M} \log M \right)^{1/5} \right) \|A y\|_{L_p(\bar{M}, \mu)}^p$$

for each $y \in Y^*$, $j = 1, 2$ and $p > 2$.

Denote for $j = 1, 2$ the injection from $L_p(K_j, \mu'_{|K_j})$ to $L_1(K_j, \mu'_{|K_j})$ by $i_{p,1}^j$ and let P be the conditional expectation projection from $L_1(\bar{K}, \mu')$ onto $J[L_1(\bar{M}, \mu)]$ following by J^{-1} . Thus

$$u^* = B P i_{p,1}^1 1_{K_1} J A + B P i_{p,1}^2 1_{K_2} J A$$

so, using definition of $\nu_q^{(n,k)}(u)$,

$$\begin{aligned} \nu_q^{(\frac{7}{8}M, 2)}(u) &\leq \sum_{j=1}^2 \nu_q^{(\frac{7}{8}M)}([P i_{p,1}^j 1_{K_j} J A]^*) \leq \\ &\leq \sum_{j=1}^2 \|1_{K_j} J A\| \|i_{p,1}^j\| \|B P\| \leq \\ &\left(\frac{1}{2} + c(p) \left(\frac{n^{p/2}}{M} \log M \right)^{1/5} \right)^{\frac{1}{p}} \|A\| \|B\| \sum_{j=1}^2 \mu'(K_j)^{\frac{1}{q}} \\ &\leq \|A\| \|B\| \left(1 + c(p) \left(\frac{n^{p/2}}{M} \log M \right)^{1/5} \right)^{\frac{1}{p}} \\ &\leq \|A\| \|B\| \left(1 + c(p) \left(\frac{n^{p/2}}{M} \log M \right)^{1/5} \right). \end{aligned}$$

This completes the proof when $p > 2$; the other case is similar. \square

Finally, we are ready to prove theorem 1.1. Without loss of generality we may assume that $\dim(Y) \leq n$. Using theorem 1.2 it is enough to prove that

$$\hat{\nu}_q^{(N)}(v) \leq (1 + \varepsilon)\nu_q^M(v),$$

for all $v : Y \rightarrow X$ and all positive integers $M \geq n$. Iterating Theorem 1.3, we get that

$$\nu_q^{([\frac{7}{8}]^k M, 2^k)}(u) \leq \prod_{j=1}^k \left(1 + c(p) \left(\frac{n^{p/2}}{[\frac{7}{8}]^{j-1} M} \log([\frac{7}{8}]^{j-1} M) \right)^{1/5} \right) \nu_q^{(M)}(u),$$

for all k such that $(\frac{7}{8})^k M \geq n$, and for all $p > 2$. The product on the right hand side of the above inequality is smaller than $1 + \varepsilon$ as long as

$$c(p) \left(\frac{n^{p/2}}{[\frac{7}{8}]^{j-1} M} \log([\frac{7}{8}]^{j-1} M) \right)^{1/5} \leq c\varepsilon.$$

Set $N = [\frac{7}{8}]^k M$, then if

$$N \geq C(p)\varepsilon^{-5}n^{p/2} \log(\varepsilon^{-5}n),$$

then

$$\hat{\nu}_q^{(N)}(u) \leq \nu_q^{(N, 2^k)}(u) \leq \nu_q^{(N)}(u).$$

This completes the proof when $p > 2$, the case $1 < p < 2$ is similar. □

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