

BRUNN-MINKOWSKI TYPE INEQUALITIES FOR THE LATTICE POINT ENUMERATOR

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ABSTRACT. Geometric and functional Brunn-Minkowski type inequalities for the lattice point enumerator $G_n(\cdot)$ are provided. In particular, we show that

$$G_n((1-\lambda)K + \lambda L + (-1, 1)^n)^{1/n} \geq (1-\lambda)G_n(K)^{1/n} + \lambda G_n(L)^{1/n}$$

for any non-empty bounded sets $K, L \subset \mathbb{R}^n$ and all $\lambda \in (0, 1)$.

We also show that these new discrete versions imply the classical results, and discuss some links with other related inequalities.

1. INTRODUCTION AND NOTATION

As usual, we write \mathbb{R}^n to represent the n -dimensional Euclidean space, and we denote by e_i the i -th canonical unit vector. We set (x, y) for the open segment with endpoints $x, y \in \mathbb{R}^n$. The n -dimensional volume of a compact set $K \subset \mathbb{R}^n$, i.e., its n -dimensional Lebesgue measure, is denoted by $\text{vol}(K)$, and as a discrete counterpart, we use $|A|$ to represent the cardinality of a finite subset $A \subset \mathbb{R}^n$.

Let \mathbb{Z}^n be the integer lattice, i.e., the lattice of all points with integral coordinates in \mathbb{R}^n . We will denote by $G_n(\cdot)$ the lattice point enumerator for the integer lattice \mathbb{Z}^n , i.e., $G_n(M) = |M \cap \mathbb{Z}^n|$. By $\lfloor x \rfloor$ we denote the floor function of the real number x , i.e., the greatest integer less than or equal to x . Similarly, $\lceil x \rceil$ represents the ceiling function of x , namely, the least integer greater than or equal to x .

Finally, given a set $M \subset \mathbb{R}^n$, χ_M represents the characteristic function of M and, moreover, we denote by $\text{int } M$, $\text{bd } M$ and $\text{cl } M$ its interior, boundary and closure, respectively. We write $M(t) = \{x \in \mathbb{R}^{n-1} : (x, t) \in M\}$ for the $(n-1)$ -dimensional section at height $t \in \mathbb{R}$ (in the direction of e_n) whereas $\pi_n(M)$ denotes the orthogonal projection of M onto $\mathbb{R}e_n$ (regarded as a

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subset of \mathbb{R}), namely $\pi_n(M) = \{t \in \mathbb{R} : M(t) \neq \emptyset\}$. Furthermore, given $r > 0$, rM stands for the set $\{rm : m \in M\}$.

Relating the volume of the Minkowski addition of two sets in terms of their volumes, one is led to the famous *Brunn-Minkowski inequality* (for extensive survey articles on this and related inequalities we refer the reader to [1, 5]). One form of it asserts that if $\lambda \in (0, 1)$ and K and L are non-empty compact subsets of \mathbb{R}^n then

$$(1.1) \quad \text{vol}((1 - \lambda)K + \lambda L)^{1/n} \geq (1 - \lambda)\text{vol}(K)^{1/n} + \lambda\text{vol}(L)^{1/n}.$$

Here $+$ is used for the Minkowski sum, i.e., $A + B = \{a + b : a \in A, b \in B\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$. Moreover, from the homogeneity of the volume, (1.1) is equivalent to

$$(1.2) \quad \text{vol}(K + L)^{1/n} \geq \text{vol}(K)^{1/n} + \text{vol}(L)^{1/n}.$$

Next we move to the discrete setting, i.e., we consider finite sets of (integer) points which are not necessarily full-dimensional unless indicated otherwise. It can easily be seen that one cannot expect to obtain a Brunn-Minkowski type inequality for the cardinality in the classical form. Indeed, simply taking $A = \{0\}$ to be the origin and any finite set $B \subset \mathbb{Z}^n$, we get

$$|A + B|^{1/n} < |A|^{1/n} + |B|^{1/n}.$$

Therefore, a discrete Brunn-Minkowski type inequality should either have a different structure or involve modifications of the sets. A first example is the simple inequality

$$(1.3) \quad |A + B| \geq |A| + |B| - 1,$$

for finite $A, B \subset \mathbb{Z}^n$ (see e.g. [16, Chapter 2]).

In [6], Gardner and Gronchi obtained an engaging discrete Brunn-Minkowski inequality: they proved that if A, B are finite subsets of the integer lattice \mathbb{Z}^n , with dimension $\dim B = n$, then

$$(1.4) \quad |A + B| \geq |D_{|A|}^B + D_{|B|}^B|.$$

Here $D_{|A|}^B, D_{|B|}^B$ are *B-initial segments*: for $m \in \mathbb{N}$, D_m^B is the set of the first m points of \mathbb{Z}_+^n in the so-called “*B-order*”, which is a particular order defined on \mathbb{Z}_+^n depending only on the cardinality of B . For a proper definition and a deep study of it we refer the reader to [6]. As consequences of (1.4) they also get two additional nice discrete Brunn-Minkowski type inequalities:

$$|A + B|^{1/n} \geq |A|^{1/n} + \frac{1}{(n!)^{1/n}} (|B| - n)^{1/n}$$

and, if $|B| \leq |A|$, then

$$|A + B| \geq |A| + (n - 1)|B| + (|A| - n)^{(n-1)/n} (|B| - n)^{1/n} - \frac{n(n - 1)}{2}.$$

These inequalities improve previous results obtained by Ruzsa in [13, 14].

An alternative to getting a “classical” Brunn-Minkowski type inequality might be to transform (one of) the sets involved in the problem. In this regard, in [9] an extension \bar{A} of A , is defined for any non-empty and finite set $A \subset \mathbb{Z}^n$ (for $n = 1$ then $\bar{A} = A \cup \{\max(A) + 1\}$, whereas for $n > 1$ the set \bar{A} is obtained by first adding the maximal cardinality section of A , and then applying the corresponding extension to every section of the latter new set). Using this technique the following discrete counterpart for (1.2) was shown:

Theorem A ([9]). *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$(1.5) \quad |\bar{A} + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

The inequality is sharp.

Aiming to get a discrete version of (1.1), it is worth noting the following: for any pair of non-empty finite sets $A, B \subset \mathbb{R}^n$, by (1.3) (and using that $|A|, |B| \geq 1$), one has

$$\begin{aligned} |(1 - \lambda)A + \lambda B| &\geq |(1 - \lambda)A| + |\lambda B| - 1 = |A| + |B| - 1 \\ &\geq (1 - \lambda)|A| + \lambda|B| \geq ((1 - \lambda)|A|^{1/n} + \lambda|B|^{1/n})^n, \end{aligned}$$

where the last inequality follows from the convexity of the function $t \mapsto t^n$, for $t \geq 0$. Nevertheless this inequality is meaningless from a geometric point of view: the point is that while the quantities $|A|, |B|$ on the right-hand side are reduced by the factors $(1 - \lambda)$ and λ , the sets $(1 - \lambda)A$ and λB on the left-hand side have the same cardinality as A and B , respectively. A possible solution would be to involve a more natural way to “count points” according to dilatations, namely, using the lattice point enumerator G_n (for compact subsets of \mathbb{R}^n) instead of the cardinality $|\cdot|$ (for finite subsets of \mathbb{R}^n).

Again, one cannot expect to obtain a Brunn-Minkowski type inequality for G_n in the classical form (1.1) (which, as we have mentioned before, would be a similar situation to what happens for $|\cdot|$ regarding a discrete version of (1.2)). Indeed, just by taking $\lambda = 1/2$, $K = [0, m - \epsilon]^n$ and $L = [0, m + \epsilon/2]^n$, with $m \in \mathbb{N}$ and $0 < \epsilon < 1$, one gets

$$(1.6) \quad G_n \left(\frac{K + L}{2} \right)^{1/n} = m < m + \frac{1}{2} = \frac{G_n(K)^{1/n} + G_n(L)^{1/n}}{2}.$$

Thus, as in (1.5), an alternative to get such an inequality for the lattice point enumerator would be to consider a certain extension of $(1 - \lambda)K + \lambda L$. So, we pose the following question:

Question 1.1. *Given compact sets $K, L \subset \mathbb{R}^n$ containing points from \mathbb{Z}^n , which is the “best” set M_λ with $(1 - \lambda)K + \lambda L \subset M_\lambda$ that ensures that*

$$G_n(M_\lambda)^{1/n} \geq (1 - \lambda)G_n(K)^{1/n} + \lambda G_n(L)^{1/n}$$

holds for a given $\lambda \in (0, 1)$?

2. MAIN RESULTS

Here we give an answer to Question 1.1, which supposes to be a discrete counterpart of (1.1):

Theorem 2.1. *Let $\lambda \in (0, 1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Then*

$$(2.1) \quad G_n((1 - \lambda)K + \lambda L + (-1, 1)^n)^{1/n} \geq (1 - \lambda)G_n(K)^{1/n} + \lambda G_n(L)^{1/n}.$$

The inequality is sharp.

We point out the following: when dealing with arbitrary non-empty subsets $K, L \subset \mathbb{R}^n$ (i.e., not necessarily bounded), from (2.1) we immediately get that

$$\begin{aligned} G_n(M_\lambda + (-1, 1)^n)^{1/n} &\geq \sup_{m \in \mathbb{N}} G_n((1 - \lambda)K_m + \lambda L_m + (-1, 1)^n)^{1/n} \\ &\geq \sup_{m \in \mathbb{N}} \left((1 - \lambda)G_n(K_m)^{1/n} + \lambda G_n(L_m)^{1/n} \right) \\ &= (1 - \lambda)G_n(K)^{1/n} + \lambda G_n(L)^{1/n} \end{aligned}$$

for any $\lambda \in (0, 1)$, where $M_\lambda = (1 - \lambda)K + \lambda L$, $K_m = K \cap [-m, m]^n$ and $L_m = L \cap [-m, m]^n$. So, for the sake of simplicity, we will present here the results in the setting of bounded sets, although they also hold in full generality.

The core ingredient in the proof of the Theorem 2.1 is its functional analogue. To introduce such a result, we present an analytical counterpart for functions of the Brunn-Minkowski inequality, the so-called *Borell-Brascamp-Lieb inequality*, originally proved in [2] and [3]. To this end, we first recall the definition of the p -mean $\mathcal{M}_p^\lambda(\cdot, \cdot)$ of two non-negative numbers, where p is a parameter varying in $\mathbb{R} \cup \{\pm\infty\}$ (for a general reference for p -means of non-negative numbers, we refer the reader to the classic text of Hardy, Littlewood, and Pólya [8] and to the handbook [4]): we consider first the case $p \in \mathbb{R}$, with $p \neq 0$: given $a, b > 0$ we set

$$\mathcal{M}_p^\lambda(a, b) = ((1 - \lambda)a^p + \lambda b^p)^{1/p}.$$

For $p = 0$, we write $\mathcal{M}_0^\lambda(a, b) = a^{1-\lambda}b^\lambda$. To complete the picture, for $p = \pm\infty$ we set $\mathcal{M}_\infty^\lambda(a, b) = \max\{a, b\}$ and $\mathcal{M}_{-\infty}^\lambda(x, y) = \min\{a, b\}$. Finally, if $ab = 0$, we define $\mathcal{M}_p^\lambda(a, b) = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$. Note that $\mathcal{M}_p^\lambda(a, b) = 0$, if $ab = 0$, is redundant for all $p \leq 0$, however it is relevant for $p > 0$. The reason to modify in this way the definition of p -mean given in [8] is due to the classical statement of the Borell-Brascamp-Lieb inequality, which is collected below. In fact, without such a modification, the thesis of the latter result would not have mathematical interest.

The following theorem (see also [5] for a detailed presentation), as previously stated, can be regarded as the functional counterpart of the Brunn-Minkowski inequality. In fact, a straightforward proof of (1.1) for compact

sets $K, L \subset \mathbb{R}^n$ of positive volume can be obtained by applying (2.2) to the characteristic functions $f = \chi_K$, $g = \chi_L$ and $h = \chi_{(1-\lambda)K+\lambda L}$ with $p = \infty$. Moreover, in the literature, the case $p = 0$ is referred to as the *Prékopa-Leindler inequality*, which supposes to be a powerful tool when dealing (among others) with *log-concave functions*.

Theorem B (The Borell-Brascamp-Lieb inequality). *Let $\lambda \in (0, 1)$. Let $-1/n \leq p \leq \infty$ and let $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be non-negative measurable functions such that*

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p^\lambda(f(x), g(y))$$

for all $x, y \in \mathbb{R}^n$. Then

$$(2.2) \quad \int_{\mathbb{R}^n} h(x) \, dx \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(\int_{\mathbb{R}^n} f(x) \, dx, \int_{\mathbb{R}^n} g(x) \, dx \right).$$

To state our main result and henceforth, we will need the following notation: for a non-negative function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, we denote by $\phi^\diamond : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ the function given by

$$\phi^\diamond(z) = \sup_{u \in (-1, 1)^n} \phi(z + u) \quad \text{for all } z \in \mathbb{R}^n.$$

Such an extension of the function ϕ is nothing but the *Asplund sum* \star (also referred to as the *sup-convolution*, which can be regarded as the functional analogue of the Minkowski sum in the setting of log-concave functions, for which we refer the reader to [15, Section 9.5] and the references therein) of the functions ϕ and $\chi_{(-1, 1)^n}$. Indeed,

$$\begin{aligned} \phi^\diamond(z) &= \sup_{u \in (-1, 1)^n} \phi(z + u) = \sup_{u \in \mathbb{R}^n} \phi(z + u) \chi_{(-1, 1)^n}(-u) \\ &= \sup_{u_1 + u_2 = z} \phi(u_1) \chi_{(-1, 1)^n}(u_2) = (\phi \star \chi_{(-1, 1)^n})(z). \end{aligned}$$

Our main result reads as follows:

Theorem 2.2. *Let $\lambda \in (0, 1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Let $-1/n \leq p \leq \infty$ and let $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p^\lambda(f(x), g(y))$$

for all $x \in K$, $y \in L$. Then

$$(2.3) \quad \sum_{z \in M \cap \mathbb{Z}^n} h^\diamond(z) \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(\sum_{x \in K \cap \mathbb{Z}^n} f(x), \sum_{y \in L \cap \mathbb{Z}^n} g(y) \right),$$

where $M = (1-\lambda)K + \lambda L + (-1, 1)^n$.

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of an n -dimensional lattice $\Lambda \subset \mathbb{R}^n$. Under the same assumptions of the above result, we may consider the auxiliary functions $f_{\mathcal{B}}, g_{\mathcal{B}}, h_{\mathcal{B}} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$f_{\mathcal{B}}(x) = f(\varphi(x)), \quad g_{\mathcal{B}}(x) = g(\varphi(x)) \quad \text{and} \quad h_{\mathcal{B}}(x) = h(\varphi(x)),$$

for all $x \in \mathbb{R}^n$, where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the linear (bijective) map given by $\varphi(x) = \sum_{i=1}^n x_i v_i$ for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Thus, as an immediate consequence of the previous result (applied to the functions $f_{\mathcal{B}}, g_{\mathcal{B}}$ and $h_{\mathcal{B}}$), we get that

$$\sum_{z \in M \cap \Lambda} h^{\diamond_{\mathcal{B}}}(z) \geq \mathcal{M}^{\lambda \frac{p}{n p + 1}} \left(\sum_{x \in K \cap \Lambda} f(x), \sum_{y \in L \cap \Lambda} g(y) \right),$$

where $M = (1 - \lambda)K + \lambda L + \phi((-1, 1)^n)$ and $h^{\diamond_{\mathcal{B}}}(z) = \sup_{u \in \phi((-1, 1)^n)} h(z + u)$ for all $z \in \mathbb{R}^n$. So, Theorem 2.2 also holds in the setting of an n -dimensional lattice $\Lambda \subset \mathbb{R}^n$.

We would like to point out that Theorem B admits an equivalent version for p -sums (instead of p -means). In this regard, in [10] a discrete version of the Borell-Brascamp-Lieb inequality for p -sums was provided. However, in contrast to the continuous setting, where one may directly obtain Theorem B from the corresponding version for p -sums (and viceversa) because of the homogeneity of the volume, one cannot expect to derive in a similar way a discrete version of Theorem B (like Theorem 2.2) from [10, Theorem 2.1]. This fact becomes clearer by comparing the corresponding discrete geometric versions, (2.1) and (1.5); the issue relies on the lack of homogeneity of our ways of measuring in the discrete setting: the cardinality and the lattice point enumerator.

In order to show Theorem 2.2, we state an auxiliary result that will allow us to get the one-dimensional case of the above-mentioned Brunn-Minkowski type inequality for the lattice point enumerator (cf. Question 1.1).

Lemma 2.1. *Let $\lambda \in (0, 1)$ and let $K, L, M \subset \mathbb{R}$ be non-empty sets such that $(1 - \lambda)K + \lambda L \subset M$. If $M = \bigcup_{i=1}^s [a_i, b_i]$ is a finite union of (pairwise disjoint) compact intervals then*

$$(2.4) \quad G_1(M) + \Delta(M) \geq (1 - \lambda)G_1(K) + \lambda G_1(L),$$

where $\Delta(M)$ denotes the number of non-integer extreme points of M , namely

$$\Delta(M) = |\{a_i \notin \mathbb{Z} : 1 \leq i \leq s\}| + |\{b_i \notin \mathbb{Z} : 1 \leq i \leq s\}|.$$

Proof. We show the result by induction on the number of intervals s of M . For the case $s = 1$, i.e., when $M = [a_1, b_1]$ is a (non-empty) compact interval, we have on the one hand that $G_1(M) = \lfloor b_1 \rfloor - \lfloor a_1 \rfloor + 1$. Moreover, denoting by $a = \inf K$, $b = \sup K$, $c = \inf L$ and $d = \sup L$, we clearly get $G_1(K) \leq G_1([a, b]) = \lfloor b \rfloor - \lfloor a \rfloor + 1$ and $G_1(L) \leq G_1([c, d]) = \lfloor d \rfloor - \lfloor c \rfloor + 1$. On the other hand, the inclusion $(1 - \lambda)K + \lambda L \subset M$ implies that

$$\lfloor b_1 \rfloor \geq b_1 - \chi_{\mathbb{R} \setminus \mathbb{Z}}(b_1) \geq (1 - \lambda) \lfloor b \rfloor + \lambda \lfloor d \rfloor - \chi_{\mathbb{R} \setminus \mathbb{Z}}(b_1)$$

and

$$-\lfloor a_1 \rfloor \geq -a_1 - \chi_{\mathbb{R} \setminus \mathbb{Z}}(a_1) \geq -(1 - \lambda) \lfloor a \rfloor - \lambda \lfloor c \rfloor - \chi_{\mathbb{R} \setminus \mathbb{Z}}(a_1),$$

and thus

$$[b_1] - [a_1] + 1 \geq (1 - \lambda)([b] - [a] + 1) + \lambda([d] - [c] + 1) - \Delta(M).$$

This, together with the above upper bounds for the lattice point enumerator G_1 of K and L , yields $G_1(M) \geq (1 - \lambda)G_1(K) + \lambda G_1(L) - \Delta(M)$, which shows the case $s = 1$.

So, we suppose that the inequality is true for $s \geq 1$ and assume that $M = \bigcup_{i=1}^{s+1} [a_i, b_i]$, where $b_i < a_{i+1}$ for all $1 \leq i \leq s$.

Denoting by $M_1 = [a_1, b_1]$ and $M_2 = \bigcup_{i=2}^{s+1} [a_i, b_i]$, we may assume, without loss of generality, that $M_1 \cap ((1 - \lambda)K + \lambda L) \neq \emptyset$ (otherwise, the result follows directly from applying the induction hypothesis with the sets K , L and M_2). Hence, we may define $m = \sup(M_1 \cap ((1 - \lambda)K + \lambda L))$ and then, since K and L are bounded (because $(1 - \lambda)K + \lambda L \subset M$), there exist $k \in \text{cl } K$ and $l \in \text{cl } L$ such that $(1 - \lambda)k + \lambda l = m$. Thus, considering the sets $K_1 = \{x \in K : x \leq k\}$, $K_2 = K \setminus K_1$, $L_1 = \{x \in L : x \leq l\}$ and $L_2 = L \setminus L_1$, we clearly have that $(1 - \lambda)K_1 + \lambda L_1 \subset M_1$ and $(1 - \lambda)K_2 + \lambda L_2 \subset M_2$. Therefore, applying the induction hypothesis (and taking into account that M_1 and M_2 are disjoint), we get

$$\begin{aligned} G_1(M) + \Delta(M) &= G_1(M_1) + G_1(M_2) + \Delta(M_1) + \Delta(M_2) \\ &\geq (1 - \lambda)(G_1(K_1) + G_1(K_2)) + \lambda(G_1(L_1) + G_1(L_2)) \\ &= (1 - \lambda)G_1(K) + \lambda G_1(L), \end{aligned}$$

as desired. \square

Remark 2.1. *One might think that if $(1 - \lambda)K + \lambda L$ is as a finite union of (pairwise disjoint) compact intervals, the set $M = (1 - \lambda)K + \lambda L$ would yield a tighter inequality in (2.4). Nevertheless, this is not true in general: if we consider $K = [-2m, -1] \cup [1, 2m]$ with $m \in \mathbb{N}$, $L = \{0\}$ and $\lambda = 1/2$, then for $M = (1 - \lambda)K + \lambda L = [-m, -1/2] \cup [1/2, m]$ we get $G_1(M) + \Delta(M) = 2m + 2$ whereas for $M' = [-m, m]$ we have $G_1(M') + \Delta(M') = 2m + 1$.*

We notice that, as shown in (1.6), the quantity $\Delta(M)$ cannot be (in general) omitted. However, we can rewrite (2.4) to provide an answer to Question 1.1 for $n = 1$, which is the one-dimensional case of Theorem 2.1:

Lemma 2.2. *Let $\lambda \in (0, 1)$ and let $K, L \subset \mathbb{R}$ be non-empty bounded sets. Then*

$$(2.5) \quad G_1((1 - \lambda)K + \lambda L + (-1, 1)) \geq (1 - \lambda)G_1(K) + \lambda G_1(L).$$

The inequality is sharp.

Proof. Let $M = \bigcup_{x \in (1 - \lambda)K + \lambda L} [[x], [x]]$. Clearly, M is a finite union of compact intervals (since K and L are bounded) containing $(1 - \lambda)K + \lambda L$. From Lemma 2.1 we then obtain $G_1(M) + \Delta(M) \geq (1 - \lambda)G_1(K) + \lambda G_1(L)$, which, together with the facts that $\Delta(M) = 0$ and $M \cap \mathbb{Z} = ((1 - \lambda)K + \lambda L + (-1, 1)) \cap \mathbb{Z}$, yields (2.5).

Finally, in order to show that the equality may be attained, we consider $K = L = [0, m]$ with $m \in \mathbb{N}$, for which $G_1((1 - \lambda)K + \lambda L + (-1, 1)) = m + 1 = (1 - \lambda)G_1(K) + \lambda G_1(L)$ for all $\lambda \in (0, 1)$. \square

Remark 2.2. *Since both sides on (2.5) remain invariant under translations by integers of the sets K and L , we may replace $(-1, 1)$ (in (2.5)) by any other interval $(m, m + 2)$, with $m \in \mathbb{Z}$.*

We note however that the solution to Question 1.1 provided by Lemma 2.2 (for $n = 1$), via $M = (1 - \lambda)K + \lambda L + (-1, 1)$, for all $\lambda \in (0, 1)$, cannot be in general improved by means of any other interval strictly contained in $(-1, 1)$. Indeed, by considering $I = [-a, 1)$, with $-1 < -a < 0$, and taking $K = [-1, 0]$, $L = [-2, 0]$ and $\lambda \in (0, 1)$ such that $\lambda + a < 1$, we get that

$$\begin{aligned} G_1((1 - \lambda)K + \lambda L + I) &= G_1([-1 - \lambda - a, 1)) = 2 \\ &< 2 + \lambda = (1 - \lambda)G_1(K) + \lambda G_1(L). \end{aligned}$$

The case $I = (-1, a]$, for $0 < a < 1$, is completely analogous and thus, no interval smaller than $(-1, 1)$ (with respect to set inclusion) can be taken into account. Furthermore, since the approach we will carry out throughout this paper relies on induction on the dimension n , the solution (to Question 1.1) given by the sum of the set $(-1, 1)$, for $n = 1$, will turn into adding $(-1, 1)^n$ for arbitrary n (see Theorem 2.1).

Now we state some auxiliary results. The following lemma can be regarded as a discrete counterpart of the well-known *Cavalieri Principle* (see [10, Lemma 3.1]).

Lemma 2.3 ([10]). *Let $\Omega \subset \mathbb{R}^n$ be a finite set, let $f : \Omega \rightarrow \mathbb{R}_{\geq 0}$ and set $f(\Omega) \subset \{k_0, k_1, \dots, k_r\}$ where $0 = k_0 < k_1 < \dots < k_r$. Then*

$$\sum_{x \in \Omega} f(x) = \sum_{i=1}^r (k_i - k_{i-1}) \left| \{x \in \Omega : f(x) \geq k_i\} \right| = \int_0^\infty \left| \{x \in \Omega : f(x) \geq t\} \right| dt.$$

We note that, under the conditions of the above result, on one hand we may assure that for any $k' \in (k_{i-1}, k_i)$,

$$\begin{aligned} (k_i - k_{i-1}) \left| \{x \in \Omega : f(x) \geq k_i\} \right| &= (k_i - k') \left| \{x \in \Omega : f(x) \geq k_i\} \right| \\ &\quad + (k' - k_{i-1}) \left| \{x \in \Omega : f(x) \geq k'\} \right|. \end{aligned}$$

On the other hand, for every $k' > k_m = \max_{x \in \Omega} f(x)$, we clearly have that $\left| \{x \in \Omega : f(x) \geq k'\} \right| = 0$. Hence, the set $\{k_0, k_1, \dots, k_r\}$ is not relevant.

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, let $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and set $f(\Omega \cap \mathbb{Z}^n) \subset \{k_0, k_1, \dots, k_r\}$ where $0 = k_0 < k_1 < \dots < k_r$. Then*

$$\sum_{x \in \Omega \cap \mathbb{Z}^n} f(x) = \sum_{i=1}^r (k_i - k_{i-1}) G_n(\{x \in \Omega : f(x) \geq k_i\}).$$

The following result yields the case $n = 1$ of Theorem 2.2 and will be used to derive (2.3).

Lemma 2.4. *Let $\lambda \in (0, 1)$ and let $\Omega_1, \Omega_2 \subset \mathbb{R}$ be non-empty bounded sets. Let $-1 \leq p \leq \infty$ and let $f, g, h : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p^\lambda(f(x), g(y))$$

for all $x \in \Omega_1, y \in \Omega_2$. Then

$$\sum_{z \in \Omega \cap \mathbb{Z}} h^\diamond(z) \geq \mathcal{M}_{\frac{p}{p+1}}^\lambda \left(\sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x), \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y) \right),$$

where $\Omega = (1 - \lambda)\Omega_1 + \lambda\Omega_2 + (-1, 1)$.

Proof. Clearly, we may assume that $\sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x), \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y) > 0$. We consider the non-negative functions $F, G, H, H^\diamond : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ given by

$$F(x) = \frac{f(x)}{a}, \quad G(y) = \frac{g(y)}{b}, \quad H(z) = \frac{h(z)}{c_p}, \quad H^\diamond(z) = \frac{h^\diamond(z)}{c_p},$$

where

$$a = \max_{x \in \Omega_1 \cap \mathbb{Z}} f(x) > 0, \quad b = \max_{y \in \Omega_2 \cap \mathbb{Z}} g(y) > 0 \quad \text{and} \quad c_p = \mathcal{M}_p^\lambda(a, b) > 0.$$

Then

$$\max_{x \in \Omega_1 \cap \mathbb{Z}} F(x) = \max_{y \in \Omega_2 \cap \mathbb{Z}} G(y) = 1.$$

First, we show that, for any $x \in \Omega_1, y \in \Omega_2$, we have that

$$(2.6) \quad H((1 - \lambda)x + \lambda y) \geq \min(F(x), G(y)).$$

To this aim, it is enough to consider $x \in \Omega_1, y \in \Omega_2$ with $f(x)g(y) > 0$. If $p \neq 0$ and $p \neq \infty$, writing $\theta = \lambda b^p / c_p^p \in (0, 1)$, we get

$$\begin{aligned} h((1 - \lambda)x + \lambda y) &\geq ((1 - \lambda)f(x)^p + \lambda g(y)^p)^{1/p} \\ &= c_p \left(\frac{(1 - \lambda)a^p F(x)^p + \lambda b^p G(y)^p}{c_p^p} \right)^{1/p} \\ &= c_p ((1 - \theta)F(x)^p + \theta G(y)^p)^{1/p} \\ &\geq c_p \min\{F(x), G(y)\}. \end{aligned}$$

For $p = 0$, we have

$$h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda = c_0 F(x)^{1-\lambda} G(y)^\lambda \geq c_0 \min\{F(x), G(y)\}.$$

For $p = \infty$, $h((1 - \lambda)x + \lambda y) \geq \max\{f(x), g(y)\} \geq c_\infty \min\{F(x), G(y)\}$ clearly holds. Therefore, we have shown (2.6).

The definition of F and G now implies that the level sets

$$\{x \in \Omega_1 : F(x) \geq t\}, \quad \{y \in \Omega_2 : G(y) \geq t\}$$

are non-empty for any $t \in [0, 1]$. Moreover, writing $\Omega_\lambda = (1 - \lambda)\Omega_1 + \lambda\Omega_2$, from (2.6) we deduce that

$$\{z \in \Omega_\lambda : H(z) \geq t\} \supset (1 - \lambda)\{x \in \Omega_1 : F(x) \geq t\} + \lambda\{y \in \Omega_2 : G(y) \geq t\}$$

and thus, by Lemma 2.2, we have

$$(2.7) \quad \begin{aligned} & \mathbf{G}_1(\{z \in \Omega_\lambda : H(z) \geq t\} + (-1, 1)) \\ & \geq (1 - \lambda)\mathbf{G}_1(\{x \in \Omega_1 : F(x) \geq t\}) + \lambda\mathbf{G}_1(\{y \in \Omega_2 : G(y) \geq t\}) \end{aligned}$$

for all $t \in [0, 1]$.

Note that, since $H^\diamond(z + u) \geq H((z + u) - u) = H(z)$ for all $u \in (-1, 1)$, we also have

$$(2.8) \quad \{z \in \Omega : H^\diamond(z) \geq t\} \supset \{z \in \Omega_\lambda : H(z) \geq t\} + (-1, 1).$$

Finally, set $\{k_0, k_1, \dots, k_r\} \supset F(\Omega_1 \cap \mathbb{Z}) \cup G(\Omega_2 \cap \mathbb{Z}) \cup H^\diamond(\Omega \cap \mathbb{Z})$, with $0 = k_0 < k_1 < \dots < k_r$ where, for some $s \in \{1, \dots, r\}$,

$$k_s = \max_{x \in \Omega_1 \cap \mathbb{Z}} F(x) = \max_{y \in \Omega_2 \cap \mathbb{Z}} G(y) = 1.$$

Then, by (2.7), (2.8) and using Corollary 2.1, we get

$$\begin{aligned} \sum_{z \in \Omega \cap \mathbb{Z}} h^\diamond(z) &= \sum_{z \in \Omega \cap \mathbb{Z}} c_p H^\diamond(z) = c_p \sum_{i=1}^r (k_i - k_{i-1}) \mathbf{G}_1(\{z \in \Omega : H^\diamond(z) \geq k_i\}) \\ &\geq c_p \sum_{i=1}^s (k_i - k_{i-1}) \mathbf{G}_1(\{z \in \Omega : H^\diamond(z) \geq k_i\}) \\ &\geq c_p \sum_{i=1}^s (k_i - k_{i-1}) \left((1 - \lambda) \mathbf{G}_1(\{x \in \Omega_1 : F(x) \geq k_i\}) \right. \\ &\quad \left. + \lambda \mathbf{G}_1(\{y \in \Omega_2 : G(y) \geq k_i\}) \right) \\ &= c_p \left((1 - \lambda) \sum_{x \in \Omega_1 \cap \mathbb{Z}} F(x) + \lambda \sum_{y \in \Omega_2 \cap \mathbb{Z}} G(y) \right) \\ &= c_p \left(\frac{1 - \lambda}{a} \sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x) + \frac{\lambda}{b} \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y) \right) \\ &\geq \mathcal{M}_{\frac{p}{p+1}}^\lambda \left(\sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x), \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y) \right). \end{aligned}$$

If $p \neq 0$ and $p \neq \infty$, the last inequality follows from the reverse Hölder inequality (see e.g. [4, Theorem 1, page 178]),

$$a_1 b_1 + a_2 b_2 \geq \left(a_1^{-p} + a_2^{-p} \right)^{-1/p} \left(b_1^q + b_2^q \right)^{1/q},$$

where $q = p/(p+1)$ is the Hölder conjugate of $(-p) \leq 1$, just by taking $a_1 = ((1-\lambda)^{1/p}a)^{-1}$, $a_2 = (\lambda^{1/p}b)^{-1}$, $b_1 = (1-\lambda)^{1/q} \sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x)$ and $b_2 = \lambda^{1/q} \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y)$.

The case $p = 0$ follows from the Arithmetic-Geometric mean inequality and the case $p = \infty$ is immediate. \square

Proof of Theorem 2.2. If $n = 1$, the result follows immediately from Lemma 2.4. Now suppose that $n > 1$ and assume that the theorem holds for dimension $n - 1$. Let $t_K \in \pi_n(K)$, $t_L \in \pi_n(L)$ and set, for the sake of brevity, $t_\lambda = (1-\lambda)t_K + \lambda t_L$. Moreover, we denote by $C_n = (-1, 1)^n$, $C_{n-1} = (-1, 1)^{n-1} \times \{0\}$, $M_\lambda = (1-\lambda)K + \lambda L$ and $M_{n-1} = (1-\lambda)K(t_K) + \lambda L(t_L) + (-1, 1)^{n-1}$. Consider the functions $f_1, g_1, h_1 : \mathbb{R}^{n-1} \rightarrow \mathbb{R}_{\geq 0}$ given by

$$f_1(x) = f(x, t_K), \quad g_1(x) = g(x, t_L), \\ h_1(x) = h(x, t_\lambda)$$

for any $x \in \mathbb{R}^{n-1}$. Since for all $x \in K(t_K)$, $y \in L(t_L)$ we have

$$h_1((1-\lambda)x + \lambda y) = h((1-\lambda)x + \lambda y, (1-\lambda)t_K + \lambda t_L) \\ \geq \mathcal{M}_p^\lambda(f(x, t_K), g(y, t_L)) = \mathcal{M}_p^\lambda(f_1(x), g_1(y)),$$

we may assert that

$$\sum_{z \in M_{n-1} \cap \mathbb{Z}^{n-1}} h_1^\diamond(z) \geq \mathcal{M}_{\frac{p}{(n-1)p+1}}^\lambda \left(\sum_{x \in K(t_K) \cap \mathbb{Z}^{n-1}} f_1(x), \sum_{y \in L(t_L) \cap \mathbb{Z}^{n-1}} g_1(y) \right).$$

This, together with the fact that

$$((1-\lambda)K + \lambda L)((1-\lambda)t_K + \lambda t_L) \supset (1-\lambda)K(t_K) + \lambda L(t_L),$$

and hence $(M_\lambda + C_{n-1})(t_\lambda) \supset M_{n-1}$, yields, in terms of f , g and h ,

$$(2.9) \quad \sum_{z \in ((M_\lambda + C_{n-1})(t_\lambda)) \cap \mathbb{Z}^{n-1}} h^\diamond(z, t_\lambda) \\ \geq \mathcal{M}_{\frac{p}{(n-1)p+1}}^\lambda \left(\sum_{x \in K(t_K) \cap \mathbb{Z}^{n-1}} f(x, t_K), \sum_{y \in L(t_L) \cap \mathbb{Z}^{n-1}} g(y, t_L) \right),$$

where $h^\diamond : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is the function given by $h^\diamond(z) = \sup_{v \in C_{n-1}} h(z+v)$, for which we have $h^\diamond(x, t_\lambda) = h_1^\diamond(x)$ for all $x \in \mathbb{R}^{n-1}$.

Now, let $f_2, g_2, h_2 : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be the functions defined by

$$f_2(t) = \sum_{x \in K(t) \cap \mathbb{Z}^{n-1}} f(x, t), \quad g_2(t) = \sum_{y \in L(t) \cap \mathbb{Z}^{n-1}} g(y, t), \quad \text{and} \\ h_2(t) = \sum_{z \in ((M_\lambda + C_{n-1})(t)) \cap \mathbb{Z}^{n-1}} h^\diamond(z, t).$$

Hence, (2.9) yields, in terms of f_2 , g_2 and h_2 ,

$$h_2((1-\lambda)t_K + \lambda t_L) \geq \mathcal{M}_{\frac{p}{(n-1)p+1}}^\lambda(f_2(t_K), g_2(t_L))$$

for any $t_K \in \pi_n(K)$, $t_L \in \pi_n(L)$, and thus we may use Lemma 2.4 with the sets $\pi_n(K)$, $\pi_n(L)$ and the functions f_2 , g_2 and h_2 to obtain

$$\sum_{t \in \Omega \cap \mathbb{Z}} h_2^\diamond(t) \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(\sum_{t_K \in \pi_n(K) \cap \mathbb{Z}} f_2(t_K), \sum_{t_L \in \pi_n(L) \cap \mathbb{Z}} g_2(t_L) \right),$$

where $\Omega = (1 - \lambda)\pi_n(K) + \lambda\pi_n(L) + (-1, 1)$. In the following we prove that $\sum_{t \in \Omega \cap \mathbb{Z}} h_2^\diamond(t) \leq \sum_{z \in M \cap \mathbb{Z}^n} h^\diamond(z)$, and hence the above inequality together with the relations

$$\begin{aligned} \sum_{t_K \in \pi_n(K) \cap \mathbb{Z}} f_2(t_K) &= \sum_{x \in K \cap \mathbb{Z}^n} f(x), \\ \sum_{t_L \in \pi_n(L) \cap \mathbb{Z}} g_2(t_L) &= \sum_{y \in L \cap \mathbb{Z}^n} g(y), \end{aligned}$$

shows the result. Indeed, from the fact that $(u, -w) \in C_n$ for any $(u, 0) \in C_{n-1}$ and $w \in (-1, 1)$, we have that $(M_\lambda + C_{n-1})(t + w) \subset M(t)$ for all $w \in (-1, 1)$ and thus we get

$$\begin{aligned} \sum_{t \in \Omega \cap \mathbb{Z}} h_2^\diamond(t) &= \sum_{t \in \Omega \cap \mathbb{Z}} \sup_{w \in (-1, 1)} h_2(t + w) \\ &= \sum_{t \in \Omega \cap \mathbb{Z}} \sup_{w \in (-1, 1)} \sum_{x \in ((M_\lambda + C_{n-1})(t + w)) \cap \mathbb{Z}^{n-1}} h^\diamond(x, t + w) \\ &\leq \sum_{t \in \Omega \cap \mathbb{Z}} \sum_{x \in M(t) \cap \mathbb{Z}^{n-1}} \sup_{w \in (-1, 1)} h^\diamond(x, t + w) \\ &= \sum_{t \in \Omega \cap \mathbb{Z}} \sum_{x \in M(t) \cap \mathbb{Z}^{n-1}} \sup_{w \in (-1, 1)} \sup_{v \in (-1, 1)^{n-1}} h(x + v, t + w) \\ &= \sum_{z \in M \cap \mathbb{Z}^n} \sup_{u \in (-1, 1)^n} h(z + u) = \sum_{z \in M \cap \mathbb{Z}^n} h^\diamond(z), \end{aligned}$$

as claimed. This finishes the proof. \square

Corollary 2.2. *Let $\lambda \in (0, 1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Let $-1/n \leq p \leq \infty$. Then*

$$(2.10) \quad \mathbf{G}_n((1 - \lambda)K + \lambda L + (-1, 1)^n) \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda (\mathbf{G}_n(K), \mathbf{G}_n(L)).$$

The inequality is sharp.

Proof. The result is an immediate consequence of Theorem 2.2, just by taking $f = \chi_K$, $g = \chi_L$ and $h = \chi_{(1-\lambda)K + \lambda L}$, for which we clearly have that $h^\diamond = \chi_{(1-\lambda)K + \lambda L + (-1, 1)^n}$.

Now, in order to show that the equality may be attained, we consider $K = L = [0, m]^n$ with $m \in \mathbb{N}$, for which $\mathbf{G}_n((1 - \lambda)K + \lambda L + (-1, 1)^n) = \mathbf{G}_n(K) = \mathbf{G}_n(L) = (m + 1)^n$. \square

We notice that (2.10) for $p = \infty$ yields (2.1) for bounded sets $K, L \subset \mathbb{R}^n$ with $G_n(K)G_n(L) > 0$. So, to prove Theorem 2.1, it is enough to deal with the case in which (only) one of the sets, say L , has no integer points. To this aim, first we show the following auxiliary result:

Theorem 2.3. *Let $K, L \subset \mathbb{R}^n$ be bounded sets such that $G_n(K)G_n(L) > 0$ and let $\alpha = m/q$ and $\beta = p/q$ with $m, p, q \in \mathbb{N}$ so that $\alpha + \beta \leq 1$. Then*

$$(2.11) \quad G_n \left(\alpha K + \beta L + \left[-\frac{q-1}{q}, \frac{q-1}{q} \right]^n \right)^{1/n} \geq \alpha G_n(K)^{1/n} + \beta G_n(L)^{1/n}.$$

Next we prove its corresponding one-dimensional case, collected in Lemma 2.5. Then, the proof of the theorem is completed in a way similar to the proof of Theorem 2.2 with the particular functions $f = \chi_K$, $g = \chi_L$ and $h = \chi_{\alpha K + \beta L}$, replacing there the use of inequality (2.5) by (2.12).

Lemma 2.5. *Let $K, L \subset \mathbb{R}$ be non-empty bounded sets and let $\alpha = m/q$ and $\beta = p/q$ with $m, p, q \in \mathbb{N}$ so that $\alpha + \beta \leq 1$. Then*

$$(2.12) \quad G_1 \left(\alpha K + \beta L + \left[-\frac{q-1}{q}, \frac{q-1}{q} \right] \right) \geq \alpha G_1(K) + \beta G_1(L).$$

Proof. First we notice that, for any $x, y \in \mathbb{R}$, we have

$$(2.13) \quad \left\lfloor \frac{mx}{q} + \frac{py}{q} + \frac{q-1}{q} \right\rfloor \geq \frac{m \lfloor x \rfloor + p \lfloor y \rfloor}{q}.$$

Indeed, given $z \in \mathbb{R}$, from the fact that

$$\frac{z}{q} + \frac{q-1}{q} \geq \frac{\lfloor z \rfloor}{q} + \frac{q-1}{q} = c + \frac{r}{q} + \frac{q-1}{q}$$

for some $c, r \in \mathbb{Z}$ with $0 \leq r \leq q-1$, we get that

$$\left\lfloor \frac{z}{q} + \frac{q-1}{q} \right\rfloor \geq \frac{\lfloor z \rfloor}{q}$$

for any $z \in \mathbb{R}$. This now implies that

$$\left\lfloor \frac{mx}{q} + \frac{py}{q} + \frac{q-1}{q} \right\rfloor \geq \frac{\lfloor mx + py \rfloor}{q} \geq \frac{m \lfloor x \rfloor + p \lfloor y \rfloor}{q},$$

which yields (2.13).

Next we show the following: if $K, L, M \subset \mathbb{R}$ are non-empty sets with $\alpha K + \beta L \subset M$ and such that $M + [-(q-1)/q, (q-1)/q] = \bigcup_{i=1}^s [a_i, b_i]$ is a finite union of (pairwise disjoint) compact intervals then

$$(2.14) \quad G_1(M + [-(q-1)/q, (q-1)/q]) \geq \alpha G_1(K) + \beta G_1(L).$$

We prove it by induction on the number of intervals s of $M + [-(q-1)/q, (q-1)/q]$. For $s = 1$, i.e., when $M + [-(q-1)/q, (q-1)/q] = [a_1, b_1]$ is a (non-empty) compact interval, we have on the one hand that $G_1(M + [-(q-1)/q, (q-1)/q]) = \lfloor b_1 + (q-1)/q \rfloor - \lceil a_1 - (q-1)/q \rceil + 1$. Moreover, denoting by $a = \inf K$, $b = \sup K$, $c = \inf L$ and $d = \sup L$, we clearly get

$G_1(K) \leq G_1([a, b]) = [b] - [a] + 1$ and $G_1(L) \leq G_1([c, d]) = [d] - [c] + 1$. On the other hand, the inclusion $\alpha K + \beta L \subset M$ implies that

$$a_1 \leq \alpha a + \beta c - \frac{q-1}{q} \leq \alpha b + \beta d + \frac{q-1}{q} \leq b_1.$$

Altogether, and using (2.13) jointly with the fact that $\lceil x \rceil = -\lfloor -x \rfloor$ for any $x \in \mathbb{R}$, we obtain

$$\begin{aligned} G_1\left(M + \left[-\frac{q-1}{q}, \frac{q-1}{q}\right]\right) &= [b_1] - [a_1] + 1 \\ &\geq \left\lfloor \alpha b + \beta d + \frac{q-1}{q} \right\rfloor - \left\lfloor \alpha a + \beta c - \frac{q-1}{q} \right\rfloor + 1 \\ &\geq \alpha [b] + \beta [d] - \alpha [a] + \beta [c] + 1 \\ &\geq \alpha G_1([a, b]) + \beta G_1([c, d]) \\ &\geq \alpha G_1(K) + \beta G_1(L). \end{aligned}$$

Thus, we suppose that (2.14) is true for $s \geq 1$ and assume that $M + [-(q-1)/q, (q-1)/q] = \bigcup_{i=1}^{s+1} [a_i, b_i]$, where $b_i < a_{i+1}$ for all $1 \leq i \leq s$.

Denoting by M_1 and M_2 the complementary subsets of M such that $M_1 + [-(q-1)/q, (q-1)/q] = [a_1, b_1]$ and $M_2 + [-(q-1)/q, (q-1)/q] = \bigcup_{i=2}^{s+1} [a_i, b_i]$, we may assume, without loss of generality, that $M_1 \cap (\alpha K + \beta L) \neq \emptyset$ (otherwise, the result follows directly from applying the induction hypothesis with the sets K , L and M_2). Hence, we may define $m = \sup(M_1 \cap (\alpha K + \beta L))$ and then, since K and L are bounded, there exist $k \in \text{cl } K$ and $l \in \text{cl } L$ such that $\alpha k + \beta l = m$. Thus, considering the sets $K_1 = \{x \in K : x \leq k\}$, $K_2 = K \setminus K_1$, $L_1 = \{x \in L : x \leq l\}$ and $L_2 = L \setminus L_1$, and taking into account that $m + [-(q-1)/q, (q-1)/q] \subset [a_1, b_1]$, we clearly have that $\alpha K_1 + \beta L_1 \subset M_1$ and $\alpha K_2 + \beta L_2 \subset M_2$. Therefore, applying the induction hypothesis (and taking into account that M_1 and M_2 are disjoint), we get

$$\begin{aligned} &G_1(M + [-(q-1)/q, (q-1)/q]) \\ &= G_1(M_1 + [-(q-1)/q, (q-1)/q]) + G_1(M_2 + [-(q-1)/q, (q-1)/q]) \\ &\geq \alpha G_1(K_1) + \beta G_1(L_1) + \alpha G_1(K_2) + \beta G_1(L_2) = \alpha G_1(K) + \beta G_1(L), \end{aligned}$$

which shows (2.14).

Now we prove (2.12). We observe that we may assume, without loss of generality, that K and L are compact. Indeed, otherwise, considering the compact sets $K' = K \cap \mathbb{Z}$ and $L' = L \cap \mathbb{Z}$, for which we have that $G_1(K') = G_1(K)$, $G_1(L') = G_1(L)$ and, from the monotonicity of $G_1(\cdot)$, $G_1(\alpha K + \beta L + [-(q-1)/q, (q-1)/q]) \geq G_1(\alpha K' + \beta L' + [-(q-1)/q, (q-1)/q])$, we would get the result. So, (2.12) follows from applying (2.14) with $M = \alpha K + \beta L$, because the fact that M is compact implies that there exists a finite sequence m_1, \dots, m_r such that $\{m_1, \dots, m_r\} + [-(q-1)/q, (q-1)/q] = M + [-(q-1)/q, (q-1)/q]$. This concludes the proof. \square

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. For the sake of brevity, we will denote by $M_\lambda = (1 - \lambda)K + \lambda L$. By Corollary 2.2 for $p = \infty$, and the monotonicity of $G_n(\cdot)$, it is enough to show the result in the case in which $K = K \cap \mathbb{Z}^n$ (for which, clearly, $G_n(K) > 0$) and $L = \{x\}$ with $x = (x_1, \dots, x_n) \notin \mathbb{Z}^n$.

First we show the case in which $\lambda = p/q$, $p, q \in \mathbb{N}$, is a rational number. Then, writing $px = z + y$, with $z \in \mathbb{Z}^n$ and $y = (y_1, \dots, y_n) \in [0, 1)^n$, and using Theorem 2.3, we have

$$\begin{aligned} G_n(M_\lambda + (-1, 1)^n)^{1/n} &= G_n\left(\frac{q-p}{p}K + \frac{1}{q}z + \prod_{i=1}^n \left(-1 + \frac{y_i}{q}, 1 + \frac{y_i}{q}\right)\right)^{1/n} \\ &\geq G_n\left(\frac{q-p}{p}K + \frac{1}{q}z + \left[-\frac{q-1}{q}, \frac{q-1}{q}\right]^n\right)^{1/n} \\ &\geq (1-\lambda)G_n(K)^{1/n} + \frac{1}{q} > (1-\lambda)G_n(K)^{1/n}, \end{aligned}$$

as desired.

Now we prove the case of an irrational $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Let I be the (possibly empty) subset of $\{1, \dots, n\}$ defined in the following way: $i \in I$ if and only if $x_i = a_i + b_i/\lambda$ for some $a_i, b_i \in \mathbb{Z}$. We point out that such $a_i, b_i \in \mathbb{Z}$ are necessarily unique, since $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Hence we may then consider the point $x' = (x'_1, \dots, x'_n)$ given by $x'_i = b_i/\lambda$ if $i \in I$ and $x'_i = 0$ otherwise, for all $i = 1, \dots, n$. First we notice that, since $\lambda x'$ is an integer point, we have

$$\begin{aligned} G_n(M_\lambda + (-1, 1)^n) &= G_n((1-\lambda)K + \lambda(x - x') + \lambda x' + (-1, 1)^n) \\ &= G_n((1-\lambda)K + \lambda(x - x') + (-1, 1)^n). \end{aligned}$$

Next, denoting by $x_0 = x - x'$, we will show that there exists $\delta > 0$ such that

$$(2.15) \quad G_n((1-\lambda)K + \lambda x_0 + (-1, 1)^n) \geq G_n((1-\mu)K + \mu x_0 + (-1, 1)^n)$$

for all μ with $|\mu - \lambda| < \delta$. Thus, taking a sequence $(r_m)_m \subset \mathbb{Q} \cap (0, 1)$ with $\lim_{m \rightarrow \infty} r_m = \lambda$ and $|r_m - \lambda| < \delta$ for all $m \in \mathbb{N}$, we get, from the previous case, that

$$\begin{aligned} G_n((1-\lambda)K + \lambda x_0 + (-1, 1)^n)^{1/n} &\geq G_n((1-r_m)K + r_m x_0 + (-1, 1)^n)^{1/n} \\ &\geq (1-r_m)G_n(K)^{1/n} + r_m G_n(\{x_0\})^{1/n} \\ &\geq (1-r_m)G_n(K)^{1/n} \end{aligned}$$

for all $m \in \mathbb{N}$, which yields the result.

To show (2.15) we notice that, since K is finite, $(1-\lambda)K + \lambda x_0 + [-1, 1]^n$ is a finite union of closed unit cubes and then, for any μ , $(1-\mu)K + \mu x_0 + [-1, 1]^n$ is the union of the corresponding translates of the cubes that constitute $(1-\lambda)K + \lambda x_0 + [-1, 1]^n$. Thus, there exists $\delta_1 > 0$ such that if $z \in \mathbb{Z}^n$ satisfies that $z \notin (1-\lambda)K + \lambda x_0 + [-1, 1]^n$ then $z \notin (1-\mu)K + \mu x_0 + (-1, 1)^n$ for all $|\mu - \lambda| < \delta_1$. Moreover, if $(1-\lambda)K + \lambda x_0 + (-1, 1)^n$ contains no boundary points, we may take $\delta = \delta_1$ and we are done.

So, we may assume that $\text{bd}((1-\lambda)K + \lambda x_0 + (-1, 1)^n) \neq \emptyset$. Let $z \in \mathbb{Z}^n$ be a boundary point of $(1-\lambda)K + \lambda x_0 + (-1, 1)^n$ and let $k \in K$. On one hand, if z is in the boundary of the cube $(1-\lambda)k + \lambda x_0 + (-1, 1)^n$, there exist $i \in \{1, \dots, n\}$ and $\varepsilon \in \{-1, 1\}$ such that $(1-\lambda)k_i + \lambda(x_i - x'_i) + \varepsilon = z_i$, where $k_i, z_i \in \mathbb{Z}$ are the i -th components of k, z , respectively. This implies that $x_i - x'_i = k_i + (z_i - \varepsilon - k_i)/\lambda$ and then, from the definition of both I and x' , we get that $i \in I$ and so $a_i = k_i + (z_i - \varepsilon - k_i)/\lambda$, which yields that $z_i - \varepsilon - k_i = 0$ (because λ is irrational) and thus $x_i - x'_i = a_i = k_i$. But then $(1-\mu)k_i + \mu(x_i - x'_i) + \varepsilon = (1-\lambda)k_i + \lambda(x_i - x'_i) + \varepsilon = z_i$ for all $\mu \in \mathbb{R}$, and hence z lies in the affine hull of a facet of the open cube $(1-\mu)k + \mu x_0 + (-1, 1)^n$, which implies that $z \notin (1-\mu)k + \mu x_0 + (-1, 1)^n$ for any μ . On the other hand, if z is not in the boundary of the cube $(1-\lambda)k + \lambda x_0 + (-1, 1)^n$ (and so not contained in it either), there exists $\delta_{z,k} > 0$ so that $z \notin (1-\mu)k + \mu x_0 + (-1, 1)^n$ for all μ with $|\mu - \lambda| < \delta_{z,k}$.

Since the number of integer points z in the boundary of $(1-\lambda)K + \lambda x_0 + (-1, 1)^n$ is finite (and K is so), we may define $\delta_2 = \min_{z,k} \delta_{z,k}$. Altogether, (2.15) holds for $\delta = \min\{\delta_1, \delta_2\}$. \square

We conclude this section by showing that the classical Borell-Brascamp-Lieb inequality (2.2) can be obtained from the discrete version (2.3) under some mild assumptions for the functions there involved:

Theorem 2.4. *The discrete Borell-Brascamp-Lieb type inequality (2.3) implies the classical Borell-Brascamp-Lieb inequality (2.2), provided that the functions f, g are Riemann integrable and h is upper semicontinuous.*

Before proving this result, we notice that it is not possible to directly obtain any of the discrete Brunn-Minkowski type inequalities stated in the introduction of the paper from the classical one (1.2), by using the method of replacing the points by suitable compact sets. As pointed out by Gardner and Gronchi in [6, pages 3996–3997],

it is worth remarking that the obvious idea of replacing the points in the two finite sets by small congruent balls and applying the classical Brunn-Minkowski inequality to the resulting compact sets is doomed to failure. The fact that the sum of two congruent balls is a ball of twice the radius introduces an extra factor of 1/2 that renders the resulting bound weaker than even the trivial bound (11) below.

We clarify that (11) in [6] coincides with (1.3) of the present paper.

In the following, for a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, we write $\phi^{\circ k} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ to denote the function given by $\phi^{\circ k}(z) = \sup_{u \in (-2^{-k}, 2^{-k})^n} \phi(z + u)$ for all $z \in \mathbb{R}^n$.

Proof. Let $m \in \mathbb{N}$ and let $K = [-m, m]^n$. We will first show that

$$(2.16) \quad \int_K h(x) dx \geq \mathcal{M}_{\frac{p}{np+1}}^\lambda \left(\int_K f(x) dx, \int_K g(x) dx \right),$$

for which we may assume (multiplying by χ_K if necessary) that f , g and h vanish outside K .

For each $k \in \mathbb{N}$, we define the functions $f_k, g_k, h_k : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ given by

$$f_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} f(z), \quad g_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} g(z)$$

and

$$h_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} h(z).$$

Writing for short $K_0 = \text{int } K$, note that for any $x, y \in K_0$ we have

$$\begin{aligned} h_k((1-\lambda)x + \lambda y) &= \inf_{z \in (1-\lambda)x + \lambda y + [0, 2^{-k}]^n} h(z) \\ &= \inf_{z \in (1-\lambda)(x + [0, 2^{-k}]^n) + \lambda(y + [0, 2^{-k}]^n)} h(z) \\ &= \inf_{z_1 \in x + [0, 2^{-k}]^n, z_2 \in y + [0, 2^{-k}]^n} h((1-\lambda)z_1 + \lambda z_2) \\ &\geq \inf_{z_1 \in x + [0, 2^{-k}]^n, z_2 \in y + [0, 2^{-k}]^n} \mathcal{M}_p^\lambda(f(z_1), g(z_2)) \\ &\geq \mathcal{M}_p^\lambda \left(\inf_{z_1 \in x + [0, 2^{-k}]^n} f(z_1), \inf_{z_2 \in y + [0, 2^{-k}]^n} g(z_2) \right) \\ &= \mathcal{M}_p^\lambda(f_k(x), g_k(y)), \end{aligned}$$

and thus, we can use Theorem 2.2 for $2^{-k}\mathbb{Z}^n$ to deduce that, for any $k \in \mathbb{N}$, we have

$$(2.17) \quad 2^{-kn} \sum_{z \in K \cap 2^{-k}\mathbb{Z}^n} (h_k)^{\diamond_k}(z) \geq \mathcal{M}_{\frac{p}{n p + 1}}^\lambda \left(2^{-kn} \sum_{x \in K_0 \cap 2^{-k}\mathbb{Z}^n} f_k(x), 2^{-kn} \sum_{y \in K_0 \cap 2^{-k}\mathbb{Z}^n} g_k(y) \right),$$

where, on the left-hand side, we have used that $(K_0 + (-2^{-k}, 2^{-k})^n) \cap 2^{-k}\mathbb{Z}^n = K \cap 2^{-k}\mathbb{Z}^n$ because $K = [-m, m]^n$.

The level sets $\{x \in K : h(x) \geq t\}$ are closed, because h is upper semi-continuous and K is closed (see [12, Theorem 1.6]), and then a standard straightforward computation shows that

$$\{x \in K : h(x) \geq t\} = \bigcap_{k=1}^{\infty} \left(\{x \in K : h(x) \geq t\} + (-2^{-k}, 2^{-k})^n \right).$$

Moreover, since h vanishes outside K , we have $\{x \in K : h(x) > t\} + (-2^{-k}, 2^{-k})^n \supset \{x \in K + [0, 2^{-k}]^n : h^{\diamond_k}(x) > t\}$ for all $t > 0$. Thus, by

using Fubini's theorem and the monotone convergence theorem, we get

$$\begin{aligned}
(2.18) \quad \int_K h(x) dx &= \int_0^\infty \text{vol}(\{x \in K : h(x) \geq t\}) dt \\
&= \int_0^\infty \text{vol} \left(\bigcap_{k=1}^\infty \left(\{x \in K : h(x) \geq t\} + (-2^{-k}, 2^{-k})^n \right) \right) dt \\
&= \int_0^\infty \lim_{k \rightarrow \infty} \text{vol} \left(\{x \in K : h(x) \geq t\} + (-2^{-k}, 2^{-k})^n \right) dt \\
&= \lim_{k \rightarrow \infty} \int_0^\infty \text{vol} \left(\{x \in K : h(x) \geq t\} + (-2^{-k}, 2^{-k})^n \right) dt \\
&\geq \lim_{k \rightarrow \infty} \int_0^\infty \text{vol}(\{x \in K + [0, 2^{-k}]^n : h^{\diamond k}(x) > t\}) dt \\
&= \lim_{k \rightarrow \infty} \int_{K+[0, 2^{-k}]^n} h^{\diamond k}(x) dx.
\end{aligned}$$

Now we show that, given $z \in \mathbb{R}^n$, $h^{\diamond k}(x) \geq (h_k)^{\diamond k}(z)$ for all $x \in z + [0, 2^{-k}]^n$. Indeed, we have

$$\begin{aligned}
h^{\diamond k}(x) &= \sup_{u \in (-2^{-k}, 2^{-k})^n} h(x+u) \geq \sup_{u \in (-2^{-k}, 2^{-k})^n} \inf_{v \in [0, 2^{-k}]^n} h(z+v+u) \\
&\geq \sup_{u \in (-2^{-k}, 2^{-k})^n} h_k(z+u) = (h_k)^{\diamond k}(z).
\end{aligned}$$

This, together with (2.18) and the fact that $K + [0, 2^{-k}]^n = K \cap 2^{-k}\mathbb{Z}^n + [0, 2^{-k}]^n$, implies that

$$\int_K h(x) dx \geq \lim_{k \rightarrow \infty} \int_{K+[0, 2^{-k}]^n} h^{\diamond k}(x) dx \geq \lim_{k \rightarrow \infty} 2^{-kn} \sum_{z \in K \cap 2^{-k}\mathbb{Z}^n} (h_k)^{\diamond k}(z).$$

Furthermore, since f is Riemann integrable and $2^{-kn} \sum_{x \in K_0 \cap 2^{-k}\mathbb{Z}^n} f_k(x)$ is a *lower sum* of $f \cdot \chi_{(-m, m]^n}$ for the partition $\{x + [0, 2^{-k}]^n \subset K : x \in 2^{-k}\mathbb{Z}^n\}$ of K , it is clear that

$$\lim_{k \rightarrow \infty} 2^{-kn} \sum_{x \in K_0 \cap 2^{-k}\mathbb{Z}^n} f_k(x) = \int_K f(x) dx.$$

Here we observe that it was crucial to work with K_0 in order to get a lower sum of $f \cdot \chi_{(-m, m]^n}$ for the above partition. We also point out the necessity of considering the characteristic function $\chi_{(-m, m]^n}$ instead of $\chi_{[-m, m]^n}$, which has no influence when computing the above integral: in this way, the function $f \cdot \chi_{(-m, m]^n}$ vanishes on the points of the corresponding facets of the cube.

The same holds for the function g and then, taking limits on both sides of (2.17), we get (2.16). Since (2.16) is true for $K = [-m, m]^n$, for every $m \in \mathbb{N}$, the proof is now concluded because

$$\int_{\mathbb{R}^n} \phi(x) dx = \lim_{m \rightarrow \infty} \int_{[-m, m]^n} \phi(x) dx,$$

for every non-negative measurable function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$. \square

It is well-known that a function is Riemann integrable if and only if it is continuous almost everywhere. Since the boundary of a convex set has null measure (and from the characterization of the upper semicontinuity in terms of the level sets) we get the following result, as a straightforward consequence of the previous one.

Corollary 2.3. *The discrete Brunn-Minkowski type inequality (2.1) implies the classical Brunn-Minkowski inequality (1.1) for bounded convex sets K and L .*

We notice the necessity of assuming convexity in the latter result: for any measurable sets $K, L \subset \mathbb{R}^n$ of positive volume, containing no rational point, one cannot expect to recover the Brunn-Minkowski inequality (1.1) with the above method of shrinking the lattice \mathbb{Z}^n , since K, L have no point in $2^{-k}\mathbb{Z}^n$, for any $k \in \mathbb{N}$.

3. RELATIONS WITH OTHER INEQUALITIES

The multiplicative version of the Brunn-Minkowski inequality is, among all its equivalent forms, the one that is naturally connected to the Prékopa-Leindler inequality (the case $p = 0$ of Theorem B). It asserts that if $\lambda \in (0, 1)$ and K and L are non-empty compact subsets of \mathbb{R}^n then

$$(3.1) \quad \text{vol}((1 - \lambda)K + \lambda L) \geq \text{vol}(K)^{1-\lambda} \text{vol}(L)^\lambda.$$

In the discrete setting, considering now non-empty bounded sets $K, L \subset \mathbb{R}^n$, from (2.10) for $p = 0$ we get

$$(3.2) \quad G_n((1 - \lambda)K + \lambda L + (-1, 1)^n) \geq G_n(K)^{1-\lambda} G_n(L)^\lambda.$$

Regarding other possible discrete versions of (3.1), we have the following engaging and elegant result, shown very recently by Halikias, Klartag and Słomka in [7] (see also [11]):

Theorem C ([7]). *Let $\lambda \in (0, 1)$ and let $f, g, h, k : \mathbb{Z}^n \rightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h(\lfloor (1 - \lambda)x + \lambda y \rfloor) k(\lceil (1 - \lambda)x + \lambda y \rceil) \geq f(x)g(y)$$

for all $x, y \in \mathbb{Z}^n$, where $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$ and $\lceil x \rceil = (\lceil x_1 \rceil, \dots, \lceil x_n \rceil)$. Then

$$\left(\sum_{x \in \mathbb{Z}^n} h(x) \right) \left(\sum_{x \in \mathbb{Z}^n} k(x) \right) \geq \left(\sum_{x \in \mathbb{Z}^n} f(x) \right) \left(\sum_{x \in \mathbb{Z}^n} g(x) \right).$$

As they observed, when applying the above result to the functions $f = \chi_K$, $g = \chi_L$, $h = \chi_{\frac{K+L}{2} + (-1, 0]^n}$ and $k = \chi_{\frac{K+L}{2} + [0, 1]^n}$, one has

$$G_n\left(\frac{K+L}{2} + (-1, 0]^n\right) G_n\left(\frac{K+L}{2} + [0, 1]^n\right) \geq G_n(K)G_n(L),$$

which yields the discrete multiplicative Brunn-Minkowski type inequality

$$(3.3) \quad G_n \left(\frac{K+L}{2} + [0, 1]^n \right) \geq \sqrt{G_n(K)G_n(L)}.$$

We notice that the sole difference between (3.3) and (3.2) (for $\lambda = 1/2$) is the necessity of adding either the closed cube of edge length 1 or the open cube of edge length 2, respectively. However, they are not comparable. Indeed, let $n = 1$ and let $K = L = [-x, x]$ with $x \in \mathbb{R}_{\geq 0}$. On one hand, for $x \in \mathbb{Z}$, we have that $G_1((K+L)/2 + [0, 1]) = 2x+2 > 2x+1 = G_1((K+L)/2 + (-1, 1))$. On the other hand, for $x \notin \mathbb{Z}$, we get $G_1((K+L)/2 + [0, 1]) = 2\lfloor x \rfloor + 2 < 2\lfloor x \rfloor + 3 = G_1((K+L)/2 + (-1, 1))$.

As pointed out in Remark 2.2, inequality (2.1) is in general not true (even for $\lambda = 1/2$) by just adding the cube $[0, 1]^n$ to the convex combination $(K+L)/2$. However, this can be solved by just considering the closed cube $[0, 1]^n$, i.e., we show that inequality (3.3) also admits a $(1/n)$ -form:

Theorem 3.1. *Let $K, L \subset \mathbb{R}^n$ be bounded sets such that $G_n(K)G_n(L) > 0$. Then*

$$(3.4) \quad G_n \left(\frac{K+L}{2} + [0, 1]^n \right)^{1/n} \geq \frac{G_n(K)^{1/n} + G_n(L)^{1/n}}{2}.$$

The inequality is sharp.

As in Theorem 2.3, we just have to prove the corresponding one-dimensional case, collected in Lemma 3.1. Then, the proof of Theorem 3.1 is completed in a way similar to the proof of Theorem 2.2 with the particular functions $f = \chi_K$, $g = \chi_L$ and $h = \chi_{\frac{K+L}{2}}$, and $\lambda = 1/2$, replacing there the use of inequality (2.5) by (3.5).

Finally, to show that the equality may be attained, we consider $K = -L = [0, m]^n$ with $m \in \mathbb{N}$ odd, for which we have $G_n((K+L)/2 + [0, 1]^n) = G_n(K) = G_n(L) = (m+1)^n$.

Lemma 3.1. *Let $K, L \subset \mathbb{R}$ be non-empty bounded sets. Then*

$$(3.5) \quad G_1 \left(\frac{K+L}{2} + [0, 1] \right) \geq \frac{G_1(K) + G_1(L)}{2}.$$

Proof. The proof is completely analogous to that of Lemma 2.5, and thus we include here just the slight differences.

First we notice that, for any $x, y \in \mathbb{R}$, we have

$$(3.6) \quad \left\lfloor \frac{x+y}{2} \right\rfloor + \frac{1}{2} \geq \frac{\lfloor x \rfloor + \lfloor y \rfloor}{2}.$$

Indeed, if $\lfloor x \rfloor + \lfloor y \rfloor$ is even then $\lfloor (x+y)/2 \rfloor = (\lfloor x \rfloor + \lfloor y \rfloor)/2$ whereas if $\lfloor x \rfloor + \lfloor y \rfloor$ is odd we get $\lfloor (x+y)/2 \rfloor \geq (\lfloor x \rfloor + \lfloor y \rfloor - 1)/2$.

Now, we notice that if $M \supset (K+L)/2$ is such that $M + [0, 1] = [a_1, b_1]$ is a (non-empty) compact interval, we have on the one hand that $G_1(M +$

$[0, 1]) = \lfloor b_1 \rfloor - \lceil a_1 \rceil + 1$. Moreover, denoting by $a = \inf K$, $b = \sup K$, $c = \inf L$ and $d = \sup L$, we clearly get $G_1(K) \leq G_1([a, b]) = \lfloor b \rfloor - \lceil a \rceil + 1$ and $G_1(L) \leq G_1([c, d]) = \lfloor d \rfloor - \lceil c \rceil + 1$. On the other hand, the inclusion $(K + L)/2 \subset M$ implies that

$$a_1 \leq \frac{a + c}{2} \leq \frac{b + d}{2} \leq b_1 - 1.$$

Altogether, and using (3.6) jointly with the fact that $\lceil x \rceil = -\lfloor -x \rfloor$ for any $x \in \mathbb{R}$, we obtain

$$\begin{aligned} G_1(M + [0, 1]) &= \lfloor b_1 \rfloor - \lceil a_1 \rceil + 1 \geq \left\lfloor \frac{b + d}{2} \right\rfloor - \left\lceil \frac{a + c}{2} \right\rceil + 2 \\ &\geq \frac{\lfloor b \rfloor + \lfloor d \rfloor}{2} - \frac{\lceil a \rceil + \lceil c \rceil}{2} + 1 = \frac{G_1([a, b]) + G_1([c, d])}{2} \\ &\geq \frac{G_1(K) + G_1(L)}{2}. \end{aligned}$$

The proof is then completed in a way analogous to the proof of Lemma 2.5. \square

We would like to point out that the corresponding version of Theorem 3.1 for an arbitrary $\lambda \in (0, 1)$ is, in general, not true. Indeed, taking $K = [0, 1]^n$, $L = [-5, 6]^n$ and $\lambda = 1/3$ one has that

$$\begin{aligned} G_n((1 - \lambda)K + \lambda L + [0, 1]^n)^{1/n} &= G_n\left(\left[\frac{-5}{3}, \frac{11}{3}\right]^n\right)^{1/n} = 5 \\ &< \frac{16}{3} = (1 - \lambda)G_n(K)^{1/n} + \lambda G_n(L)^{1/n}. \end{aligned}$$

We collect here an open question that arose during our study:

Question 3.1. *Regarding the statement of Theorem 3.1, is the assumption $G_n(K)G_n(L) > 0$ necessary?*

We conclude the paper by proving that inequality (1.5) admits a version in the spirit of (2.1), namely that one may add another (fixed) set to the Minkowski sum $A + B$ instead of considering the extension \bar{A} of A . We show that an appropriate set to be taken into account in this respect is the lattice cube $\{0, 1\}^n$, which also fits well with inequality (3.4).

Theorem 3.2. *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$(3.7) \quad |A + B + \{0, 1\}^n|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

The inequality is sharp.

Proof. The proof proceeds by (finite) induction on the dimension n . Since it follows similar steps to the proof of Theorem A (see [9, Theorem 2.1]), we include here just the slight differences with respect to the argument used therein.

The case $n = 1$ is a direct consequence of (1.3):

$$|A + B + \{0, 1\}| \geq |A| + |B| + |\{0, 1\}| - 2 = |A| + |B|.$$

For the inductive step, we define

$$c_A = \max_{m \in \mathbb{Z}} |A(m)| > 0, \quad c_B = \max_{m \in \mathbb{Z}} |B(m)| > 0, \quad c = \left(c_A^{1/(n-1)} + c_B^{1/(n-1)} \right)^{n-1}$$

as well as the functions defined on \mathbb{Z} by $f_A(m) = |A(m)|/c_A$, $f_B(m) = |B(m)|/c_B$,

$$f_{A+B+\{0,1\}^n}(m) = \frac{|(A + B + \{0, 1\}^n)(m)|}{c}$$

and

$$f_{A+B+(\{0,1\}^{n-1} \times \{0\})}(m) = \frac{\left| \left(A + B + (\{0, 1\}^{n-1} \times \{0\}) \right) (m) \right|}{c}.$$

Thus, by the induction hypothesis, the relation

$$\begin{aligned} & \left| \left(A + B + (\{0, 1\}^{n-1} \times \{0\}) \right) (m_1 + m_2) \right| \\ & \geq |A(m_1) + B(m_2) + \{0, 1\}^{n-1}| \geq \left(|A(m_1)|^{1/(n-1)} + |B(m_2)|^{1/(n-1)} \right)^{n-1} \end{aligned}$$

holds for all $m_1 \in \pi_n(A)$, $m_2 \in \pi_n(B)$. Now, by an argument analogous to that in [9, p. 1849] (cf. [9, (4.3)]), we get that

$$\begin{aligned} \{m \in \mathbb{Z} : f_{A+B+(\{0,1\}^{n-1} \times \{0\})}(m) \geq t\} & \supset \{m \in \mathbb{Z} : f_A(m) \geq t\} \\ & \quad + \{m \in \mathbb{Z} : f_B(m) \geq t\} \end{aligned}$$

for all $t \in [0, 1]$. This, and using (3.7) for $n = 1$ together with the fact that

$$\{m \in \mathbb{Z} : f_{A+B+\{0,1\}^n}(m) \geq t\} \supset \{m \in \mathbb{Z} : f_{A+B+(\{0,1\}^{n-1} \times \{0\})}(m) \geq t\} + \{0, 1\},$$

allows us to conclude the proof of (3.7) by following the same steps to those in the proof of Theorem A. Finally, to show that the inequality is sharp, we consider the lattice cubes $A = \{0, m_1\}^n$ and $B = \{0, m_2\}^n$, for $m_1, m_2 \in \mathbb{Z}$. \square

Since $A + (\{0\}^{n-1} \times \{0, 1\})$ contains at least the same amount of points that the union of A and its maximal cardinality section, and taking into account that $\{0, 1\}^n = (\{0, 1\}^{n-1} \times \{0\}) + (\{0\}^{n-1} \times \{0, 1\})$, from the definition of \bar{A} is then immediate that $|\bar{A}| \leq |A + \{0, 1\}^n|$. However, inequalities (1.5) and (3.7) are not comparable. Indeed, if we consider on one hand $A = \{(0, 0), (1, 0), (2, 0), (1, 1)\}$ and $B = \{0, 1\}^2$, we have that $|\bar{A}| = 10 = |A + \{0, 1\}^2|$ (although $\bar{A} \neq A + \{0, 1\}^2$) but $|\bar{A} + B| = 20 > 18 = |A + \{0, 1\}^2 + B|$ (see Figure 3). On the other hand, for $A = \{(0, 0), (0, 1), (1, 1), (4, 1)\}$ and $B = \{0, 1\}^2$ we obtain $|\bar{A} + B| = 21 < 24 = |A + \{0, 1\}^2 + B|$.

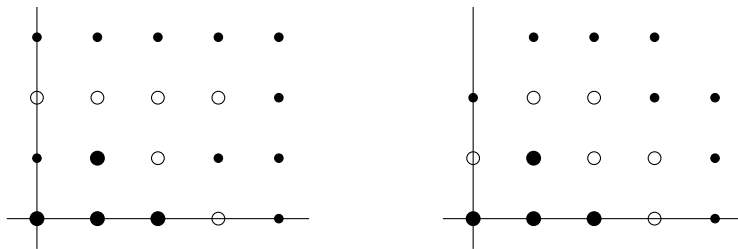


FIGURE 1. Left: the sets A (thick points), \bar{A} (thick and hollow points) and $\bar{A} + B$. Right: the sets A (thick points), $A + \{0, 1\}^2$ (thick and hollow points) and $A + \{0, 1\}^2 + B$.

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