# TOPICS IN PROBABILITY THEORY AND STOCHASTIC PROCESSES
**Home Work 7, due on THURSDAY October 30,**
**Instructor: Prof. Artem Zvavitch**
**This is to prepare your for exam. Please, TRY to do it before November 4, do 5 problems to get 20pts, do 10 problem to get 40pts**

**Problem 1.** *Consider exponential random variable $X$ with rate $\lambda$ and independent exponential random variable $Y$ with rate $\mu$, please find:*

- *Moment generating function, i.e.* $\phi(t) = \mathbb{E}e^{tX}$,
- $\mathbb{E}(X+Y)^2$
- $Var(X)$,
- $\mathbb{E}[X|X<1]$
- $\Pr(X<Y)$,
- *Show that conditional density function of $X$, given $X+Y=t$ is*
$$f_{(X|X+Y)}(x|t) = \frac{(\lambda-\mu)e^{-(\lambda-\mu)x}}{1-e^{-(\lambda-\mu)x}}, \text{ for } x \in (0,t).$$
- *Find* $\mathbb{E}[X+Y=t]$.

**Problem 2.** *Specify the classes of the following Markov chain and determine whether they are transient or recurrent:*

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

*Find* $\mathbf{P_T}$ *(transition probabilities from transient state to transient state). Find an expected number of times the chain will be in state 4 given that it starts at state 3? What is the probability that the chain ever comes to state 3 starting at state 4? What is the probability that the chain ever comes to state 3 starting at state 1?*

**Problem 3.** *Consoder the following Markov chain.*

$$\mathbf{P} = \begin{pmatrix} .4 & .5 & .1 \\ .3 & .2 & .5 \\ .1 & .6 & .3 \end{pmatrix}$$

*Find* $\Pr(X_6 = 2|X_4 = 3)$, *also in the long run, what proportion of time is the process in each of the three states?*

**Problem 4.** *Please find* $\pi_0$ *for the branching processes when* $P_0 = 1/12, P_1 = 1/4, P_2 = 2/3$.

**Problem 5.** *Let* $\{N(t), t \geq 0\}$ *be a Poisson process with rate* $\mu$. *Let* $S_n$ *denote the time of the* $n$*th event. Find*

- $\mathbb{E}S_3$,
- $\mathbb{E}[S_3|N(1) = 2]$,
- $\mathbb{E}[N(3) - N(1)|N(1) = 3]$.

**Problem 6.** *Cars pass a certain street location according to a Poisson process with rate* $\lambda$. *A woman who wants to cross the street at that location waits until she can see that no cars will come by in next* $T$ *time units. Find the probability that her waiting time is* $0$. *Find her expected waiting time (Hint: condition on the time of the first car).*

**Problem 7.** *Men and women enter a supermarket according to independent Poisson processes having respective rates two and four per minute. Starting at an arbitrary time, compute the probability that at least two men arrive before three women arrive.*

**Problem 8.** *Consider a birth and death process with birth rates* $\lambda_i = i\lambda$, $i \geq 0$, *and death rates* $\mu_i = (i+1)\mu$, $i > 0$. *Determine the expected time to go from state* $0$ *to state* $4$.

**Problem 9.** *Consider two machinex, both of which have an exponential lifetime with mean* $1/\lambda$. *There is a single repairman that can service machines at exponential rate* $\mu$. *Set up Kolmogorov backward equation.*

**Problem 10.** *A small barbershop, operated by a single barber, has a room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean* $1/4$ *hour. What is the average number of customers in the shop? The proportion of potential customers that enter the shop? If the barber could work twice as fast, how much more business would he do?*